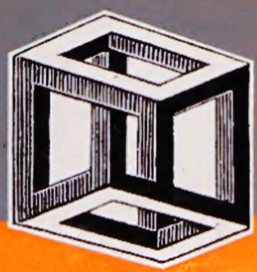
D. S. Mitrinović

Functions of a Complex Variable



Tutorial Texts

D. S. MITRINOVIĆ • FUNCTIONS OF A COMPLEX VARIABLE

TUTORIAL TEXT No. 2

FUNCTIONS OF A COMPLEX VARIABLE

BY

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FOREWORD

The problems contained in this series have been collected over many years with the aim of providing students and teachers with material, the search for which would otherwise occupy much valuable time. Hitherto this concentrated material has only been accessible to the very restricted public able to read Serbian*.

I greatly welcome the appearance of the Tutorial Texts based on my problem collection. Cooperation with colleagues in Australia and the U.S.A. and their initiative have made it possible. For the preparation of this Text, I wish to thank E. S. Barnes and J. R. M. Radok.

D. S. MITRINOVIĆ.

Belgrade, November 1964.

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INTRODUCTION

This text deals with elementary properties and applications of functions of a complex variable. The use of the calculus of residues for the evaluation of real integrals is to be treated in Tutorial Text 4.

For an account of the theory of complex numbers and functions, the reader may refer to one of the many standard texts on this subject, e.g.,

L. V. AHLFORS: "Complex Analysis, An Introduction to the Theory of Analytic Functions of One Complex Variable", McGraw-Hill, New York, 1953.

H. BEHNKE–F. SOMMER: "Theorie der analytischen Funktionen einer Komplexen Veränderlichen", Grundlehren der mathematischen Wissenschaften, Band LXXVII, Springer, Berlin—Göttingen—Heidelberg, 2d ed. 1962.

L. BIEBERBACH: "Einführung in die Funktionentheorie," 2nd ed. Wissenschaft and Fachbuch, Bielefeld, 1952.

R. V. CHURCHILL: "Introduction to Complex Variables and Applications", 1st ed. McGraw-Hill, New York, 1948.

E. T. COPSON: "An introduction to the Theory of Functions of a Complex Variable", Clarendon Press, Oxford, 1935.

G. SANSONE–J. GERRETSEN: "Lectures on the theory of functions of a complex variable", Vol. I, P. Noordhoff, Groningen, 1960.

E. GOURSAT: "Functions of a Complex Variable", Vol. II, Part One, Dover Publications Inc., New York, 1959.

K. KNOPP: "Elements of the Theory of Functions", Dover, New York, 1953.

K. KNOPP: "Problem Book in the Theory of Functions", Dover, New York, 1948—1953.

M. A. LAVRENT'EV and B. V. SHABAT: "Methods of the Theory of Functions of a Complex Variable", 2nd. ed. Gosudarstv., Izdat. Fiz-Mat. Lit. Moscow, 1958.

E. G. PHILLIPS: "Functions of a Complex Variable, with Applications", 8th ed., Oliver and Boyd, Edinburgh, 1957.

I. I. PRIVALOV: "Introduction to the Theory of Functions of a Complex Variable", 9th ed. Gosudarstv., Izdat. Teh-Teor. Lit. Moscow, 1954.

E. C. TITCHMARSCH: "The Theory of Functions", 2nd ed. Oxford Univ. Press, London, 1939.

As a convenience to the student, we give here a brief outline of this theory, insofar as it relates to most of the problems in the following chapters, and introduce the notations used in the text. Throughout, unless stated otherwise, all numbers are assumed to be complex.

§ 0.1. Complex Numbers

The field C of complex numbers is usually built up from the real number system by adjoining a root of the equation $x^2+1=0$, or by considering ordered pairs of real numbers with appropriate definitions of addition and multiplication. In either case, we are led to the specification of a complex number in the form

$$x+iy,$$

where x and y are real numbers and i is the "imaginary unit". The rules of manipulation are precisely the same as for real numbers, provided that we define i^2 to be equal to -1.

Thus, if $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$ are complex numbers (x_1, y_1, x_2, y_2 real), then

$$\begin{aligned} z_1+z_2 &= (x_1+x_2)+i(y_1+y_2), \\ z_1z_2 &= x_1x_2+ix_1y_2+ix_2y_1+i^2y_1y_2 \\ &= (x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1). \end{aligned}$$

Under these rules, the set of all complex numbers $x+iy$, with x, y real, is easily shown to be a field, whose zero (additive identity) is $0+i0$ (usually written as 0) and whose unity (multiplicative identity) is $1+i0$ (usually written as 1). In other words, for any complex numbers z_1, z_2, z_3, we have

$$\begin{aligned} &z_1+z_2 = z_2+z_1, && z_1z_2 = z_2z_1, \\ &z_1+(z_2+z_3)=(z_1+z_2)+z_3, && z_1(z_2z_3)=(z_1z_2)z_3, \\ &z_1+0 = 0+z_1 = z_1, && z_1\cdot 1 = 1\cdot z_1 = z_1, \end{aligned}$$

$$z_1(z_2+z_3) = z_1z_2+z_1z_3;$$

if $z_1 = x_1+iy_1$, then $-z_1$, defined to be $-x_1+i(-y_1)$ satisfies

$$z_1+(-z_1) = 0;$$

if $z_1 \neq 0$, then $1/z_1$, defined by

$$\frac{1}{z_1} = \frac{x_1}{x_1^2+y_1^2} + i\frac{-y_1}{x_1^2+y_1^2},$$

satisfies

$$z_1 \cdot \frac{1}{z_1} = 1.$$

The real numbers x, y, where $z = x+iy$, are called the *real* and *imaginary parts* of z, and we write

$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

The system of complex numbers with zero imaginary part is isomorphic to the real number system. Hence, if $\operatorname{Im} z = 0$, we say that z is *real*; if $\operatorname{Re} z = 0$, and $z \neq 0$, we say that z is *purely imaginary*.

The *conjugate* of $z = x+iy$, denoted by $\bar{z}$, is defined by

$$\bar{z} = x-iy.$$

We then see that

$$z\bar{z} = x^2+y^2$$

is a non-negative real number. Also,

$$\operatorname{Re} z = x = \tfrac{1}{2}(z+\bar{z}),$$
$$\operatorname{Im} z = y = \frac{1}{2i}(z-\bar{z}).$$

It is easily verified that for all z_1, z_2

$$\bar{\bar{z}}_1 = z_1, \quad \overline{z_1+z_2} = \bar{z}_1+\bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \ (z_2 \neq 0).$$

The *modulus* (absolute value) of $z = x+iy$, denoted by $|z|$, is defined by

$$|z| = \sqrt{(z\bar{z})} = \sqrt{(x^2+y^2)};$$

thus $|z|$ is a non-negative real number, and $z = 0$ if and only if $|z| = 0$. Using the above rules for conjugates, it is easy to show that for all z_1, z_2

$$|\bar{z}_1|=|z_1|,\ |z_1z_2|=|z_1|\cdot|z_2|,\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\quad(z_2\neq 0).$$

The inequalities for sums and differences have the same form as for real numbers, namely

$$|z_1+z_2|\leqq|z_1|+|z_2|,$$
$$|z_1-z_2|\geqq|z_1|-|z_2|.$$

If z is real, then the definition of $|z|$ agrees with that for real numbers; for any z, we have

$$|\mathrm{Re}\,z|\leqq|z|,\qquad|\mathrm{Im}\,z|\leqq|z|.$$

§ 0.2. The Argand Plane

Consider a Cartesian plane with rectangular axes OX, OY. With any complex number $z=x+iy$ (x, y real) we may associate

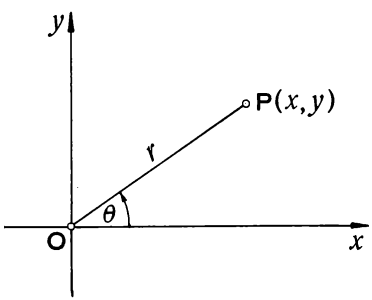

Fig. 1

the point P of the plane with coordinates (x, y) referred to the given axes. In this way, we establish a one-one correspondence $z\leftrightarrow P$, between C and the Cartesian plane, in which

$$\mathrm{Re}\,z\leftrightarrow\text{abscissa of }P,$$
$$\mathrm{Im}\,z\leftrightarrow\text{ordinate of }P,$$
$$|z|=\sqrt{(x^2+y^2)}=|\overrightarrow{OP}|.$$

Alternatively, since any point P uniquely determines its

position vector $\overrightarrow{OP}$, we have a one-one correspondence between $z = x+iy \in C$, and the vector $\overrightarrow{OP}$ with components x, y. This correspondence is preserved under addition: i.e., if $z_1 \leftrightarrow \overrightarrow{OP}_1$ and $z_2 \leftrightarrow \overrightarrow{OP}_2$, then

$$z_1+z_2 \leftrightarrow \overrightarrow{OP}_1+\overrightarrow{OP}_2.$$

Note that if $z \leftrightarrow P$, then $\bar{z} \leftrightarrow \bar{P}$, where $\bar{P}$ is the reflection of P in the x-axis.

By considering polar coordinates in the plane, we are led to the *polar representation* of complex numbers. Let $z \leftrightarrow P$, where P has polar coordinates (r, θ); then, since

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have

$$z = x+iy = r(\cos\theta+i\sin\theta).$$

Here $r = |z|$, and θ is determined up to a multiple of 2π by the relations

$$\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}.$$

θ is called the *argument* (amplitude, arcus) of z, written $\arg z$; it is defined for all $z \neq 0$.

In order to specify a unique value of $\arg z$, we may restrict its values to some interval of length 2π. The most common convention is that

$$-\pi < \arg z \leqq \pi;$$

the unique argument satisfying this inequality is then called the *principal value* of the argument, sometimes denoted by $\operatorname{Arg} z$.

The expression $\cos\theta+i\sin\theta$ is sometimes denoted by $\operatorname{cis}\theta$; thus

$$z = r\operatorname{cis}\theta, \text{ where } r = |z|, \ \theta = \arg z.$$

By multiplying out, we may show that

$$\operatorname{cis}\theta_1 \operatorname{cis}\theta_2 = \operatorname{cis}(\theta_1+\theta_2).$$

Hence, if $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis} (\theta_1 + \theta_2);$$

it follows that

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|,$$

as before, and also that

$$\arg (z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2.$$

This last relation is not necessarily true for principal values of the argument; it asserts only that $\arg z_1 + \arg z_2$ is one of the values of $\arg (z_1 z_2)$.

From the multiplicative property of $\operatorname{cis} \theta$ it may be shown by induction that, for any integer n, $(\operatorname{cis} \theta)^n = \operatorname{cis} n\theta$, i.e., that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad (n = 0, \pm 1, \pm 2, \ldots).$$

This result is known as De Moivre's Theorem.

It is more usual to write $\cos \theta + i \sin \theta$ as $e^{i\theta}$ or $\exp i\theta$. This exponential notation is justified by the fact that the "laws of indices" continue to hold; i.e., if θ_1, θ_2 are real, then

$$e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}$$

and

$$(e^{i\theta})^n = e^{ni\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Thus $e^{i\theta}$ appears as a periodic (complex) function of the real variable θ, with period 2π, since

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1.$$

From the relations

$$e^{i\theta} = \cos \theta + i \sin \theta,$$
$$e^{-i\theta} = \cos (-\theta) + i \sin (-\theta) = \cos \theta - i \sin \theta$$

we obtain Euler's formulae

$$\cos \theta = \tfrac{1}{2} (e^{i\theta} + e^{-i\theta}), \quad \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \quad (\theta \text{ real}).$$

§ 0.3. Applications to Plane Geometry

Using the correspondence described in § 0.2, we may formulate problems of plane geometry in terms of complex numbers. In this connection, it is usual to abbreviate "the point P corresponding to z" simply to "the point z". We have only to interpret Cartesian coordinates (x, y) as the real and imaginary parts of a complex number $z = x+iy$.

Thus the locus of points z satisfying the relation

(i) $$\alpha z+\bar{\alpha}\bar{z}+c = 0 \quad (c \text{ real}, \alpha \neq 0)$$

is a straight line,

(ii) $$az\bar{z}+\beta z+\bar{\beta}\bar{z}+c = 0 \qquad (a, c \text{ real}, a \neq 0)$$

is a circle,

(iii) $$|z-\alpha| = r \qquad (r > 0)$$

is a circle, centre α, radius r,

(iv) $$\left|\frac{z-\alpha}{z-\beta}\right| = k \qquad (k > 0)$$

is a circle belonging to the coaxial system with limiting points α and β (a straight line bisecting the line segment joining α and β if $k = 1$),

(v) $$\arg\left(\frac{z-\alpha}{z-\beta}\right) = \theta \qquad (\theta \text{ real})$$

is a circular arc with end-points α and β.

A curve γ in the plane may be specified by the relations

$$\left.\begin{aligned} x &= f(t) \\ y &= g(t) \end{aligned}\right\} \quad (a \leqq t \leqq b),$$

where f, g are continuous real functions. These may be written as the single relation

$$z = \varphi(t) = f(t)+ig(t) \qquad (a \leqq t \leqq b),$$

where now φ is a continuous complex function of the real variable t.

For example, the unit circle, described once in the positive direction, may be specified by

$$z = e^{it} \qquad (0 \leqq t < 2\pi).$$

If here f and g are differentiable functions and we define

$$z'(t) = f'(t) + ig'(t) = \frac{dx}{dt} + i\frac{dy}{dt},$$

then, at the point $z_0 = z(t_0)$, γ has a tangent whose direction is given by

$$\theta = \arg z'(t_0), \text{ provided } z'(t_0) \neq 0.$$

§ 0.4. Polynomials

The simplest complex function is the polynomial of the form

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n$ are arbitrary complex numbers. $f(z)$ is defined for all z; it is said to have degree n, if $a_n \neq 0$. (No degree is assigned to the zero polynomial; alternatively, with suitable conventions, we may say that its degree is $-\infty$.)

A *zero* of f is a number α satisfying $f(\alpha) = 0$, and $f(\alpha) = 0$ if and only if

$$f(z) = (z-\alpha)g(z)$$

for some polynomial g. If

$$f(z) = (z-\alpha)^r h(z), \qquad h(\alpha) \neq 0,$$

where $r \geqq 1$ and h is a polynomial, α is called a *zero* of f of *multiplicity* r (a simple zero if $r = 1$, a double zero if $r = 2$, etc.).

A celebrated theorem of Gauss asserts that a polynomial of degree $n \geqq 1$ has a zero. From this it follows that a polynomial f of degree n has precisely n zeros, if each zero is counted according to its multiplicity. If the zeros are $\alpha_1, \ldots, \alpha_k$, of multiplicities $r_1, \ldots, r_k$, then

$$f(z) \equiv a_n(z-\alpha_1)^{r_1} \ldots (z-\alpha_k)^{r_k}, \quad r_1 + \ldots + r_k = n.$$

The zeros of $f(z) = z^n - 1$ are the n-th roots of unity

$$\exp\left(\frac{2\pi i r}{n}\right) \qquad (r = 0, 1, \ldots, n-1);$$

on the Argand plane they are the vertices of a regular n-gon inscribed in the unit circle.

More generally, if $\beta = \rho\, e^{i\varphi}$ $(\rho = |\beta|)$, then the n zeros of $z^n - \beta$ are

$$\rho^{\frac{1}{n}} \exp\left(\frac{i\varphi + 2\pi i r}{n}\right) \qquad (r = 0, 1, \ldots, n-1).$$

§ 0.5. The Elementary Complex Functions

Exponential. For $z = x + iy$ (x, y real), we define

$$e^z = \exp z = e^x (\cos y + i \sin y);$$

this definition accords with the real exponential function when z is real, and with our definition in § 0.2 when z is purely imaginary. From this definition it follows at once that for all z_1, z_2

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2};$$

in particular, $e^z \cdot e^{-z} = e^0 = 1$, for all z, whence we deduce that e^z is never zero. Further,

$$\overline{\exp z} = \exp \bar{z}, \qquad |\exp z| = \exp x.$$

The equation

$$\exp z = 1$$

is equivalent to the real equations

$$|\exp z| = \exp x = 1, \text{ whence } x = 0,$$

and

$$\cos y = 1, \ \sin y = 0, \text{ whence } y = 2n\pi$$

for some integer n; hence all solutions are given by

$$z = 2n\pi i \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Trigonometric. We define, for all z,

$$\cos z = \tfrac{1}{2}(e^{iz}+e^{-iz}), \quad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz});$$

the other trigonometric functions are defined in the usual way as quotients or reciprocals of these. The standard trigonometrical identities are easily deduced. It also follows easily that

$$\overline{\cos z} = \cos \bar{z}, \quad \overline{\sin z} = \sin \bar{z},$$

whence

$$\begin{aligned} |\cos z|^2 = \cos z \cos \bar{z} &= \tfrac{1}{2}\{\cos(z+\bar{z})+\cos(z-\bar{z})\} \\ &= \tfrac{1}{2}(\cos 2x+\cosh 2y) \end{aligned}$$

and, similarly,

$$|\sin z|^2 = \tfrac{1}{2}(\cosh 2y-\cos 2x).$$

The only zeros of the trigonometric functions are their real zeros, namely

$$\begin{aligned} &\sin z \text{ and } \tan z\text{: } z = n\pi \quad (n = 0, \pm 1, \pm 2, \ldots), \\ &\cos z\text{: } z = (n+\tfrac{1}{2})\pi \quad (n = 0, \pm 1, \pm 2, \ldots). \end{aligned}$$

Hyperbolic. The hyperbolic functions are defined as in real analysis:

$$\cosh z = \tfrac{1}{2}(e^{z}+e^{-z}), \quad \sinh z = \tfrac{1}{2}(e^{z}-e^{-z}).$$

Hence

$$\cosh z = \cos iz, \quad \sinh z = -i \sin iz.$$

Logarithm. We say that w is a logarithm of z, $w = \log z$, if $\exp w = z$. For any $z \neq 0$, this relation yields infinitely many logarithms of z, namely

$$w = \log |z| + i \arg z,$$

where log denotes here the real natural logarithm, and $\arg z$ is any value of the argument of z.

The principal value of the logarithm, sometimes denoted by Log (or ln), corresponds to the principal value of the argument, i.e.,

$$\operatorname{Log} z = \log |z| + i \operatorname{Arg} z,$$

where

$$-\pi < \operatorname{Arg} z \leqq \pi.$$

The multiplicative property

$$\log (z_1 z_2) = \log z_1 + \log z_2 \quad (z_1 \neq 0,\ z_2 \neq 0)$$

holds in the sense that, for any choice of $\log z_1$ and $\log z_2$, the right-hand side is a value of $\log (z_1 z_2)$.

Powers. If z, α are complex numbers with $z \neq 0$, z^α has several determinations given by

$$z^\alpha = \exp (\alpha \log z);$$

exceptionally, when $z = e$, z^α is uniquely specified as $\exp \alpha$. Otherwise, z^α is uniquely defined only when α is an integer, and the definition then agrees with the usual definition of powers and reciprocals. If α is not an integer, the principal value of z^α is given by choosing the principal value of the logarithm, i.e.,

$$\exp (\alpha \operatorname{Log} z).$$

§ 0.6. Continuity and Differentiability

If f is a (complex-valued) function and $f(z)$ is defined for all z in a neighbourhood of α, except perhaps at α, then we say that

$$\lim_{z \to \alpha} f(z) = \lambda,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - \lambda| < \varepsilon$$

for all z with

$$0 < |z - \alpha| < \delta.$$

If

$$\lim_{z \to \alpha} f(z) = f(\alpha),$$

f is said to be *continuous* at α.

If S is any set of points, f is said to be continuous on S if it is continuous at each point of S. As in real analysis, the sum, product and composition of continuous functions are continuous; so also is the quotient of continuous functions, on any set on which the denominator does not vanish.

All polynomials, $\sin z$, $\cos z$ and $\exp z$ are continuous everywhere; rational functions and the remaining trigonometric functions are continuous wherever they are defined. $\operatorname{Arg} z$ and $\operatorname{Log} z$ are continuous on the domain formed by removing from the plane the negative real axis ($y = 0$, $-\infty < x \leqq 0$).

A function f is said to be *differentiable* at α if

$$\lim_{z \to \alpha} \frac{f(z)-f(\alpha)}{z-\alpha}$$

exists; the limit is then denoted by $f'(\alpha)$. The usual rules hold for the differentiation of the sum, product, quotient and composition of differentiable functions.

If D is a domain (i.e., an open connected set of points in the plane), f is said to be *regular* (analytic, holomorphic) on D if it is differentiable at each point of D. If S is an arbitrary set of points, f is said to be regular on S if it is regular on some domain containing S. In particular, f is regular at a point z_0 if it is regular in some disk with centre z_0.

Let $u(x, y)$ and $v(x, y)$ be the real and imaginary parts of $f(z)$ ($z = x+iy$), so that we may write

$$w = u+iv = f(z) = f(x+iy).$$

Then, if f is differentiable at $z_0 = x_0+iy_0$, the functions u, v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at the point (x_0, y_0). We then have

$$\begin{aligned} f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0)+i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{1}{i}\left\{\frac{\partial u}{\partial y}(x_0, y_0)+i\frac{\partial v}{\partial y}(x_0, y_0)\right\}. \end{aligned}$$

Conversely, if the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ exist, are contenuous, and satisfy the Cauchy-Riemann equations at each point of a domain D, then f is regular on D. Then $u(x, y)$ and $v(x, y)$ are solutions of Laplace's equation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0,$$

i.e., they are harmonic functions on D, and the functions u and v are said to be *conjugate*; each is determined by the other to within an additive constant.

All polynomials and the functions $\exp z$, $\sin z$, $\cos z$, $\sinh z$ and $\cosh z$ are regular on the whole (finite) plane. Rational functions and the remaining trigonometric and hyperbolic functions are regular at all points at which they are defined. The principal branches of $\log z$ and z^α (α not integral) are regular on the domain formed by removing from the plane the negative real axis, on which $\arg z$ satisfies: $-\pi < \arg z \leqq \pi$.

§ 0.7. Taylor and Laurent Series

If the function f is regular at the point α, then, in a neighbourhood of α, $f(z)$ may be expressed as a power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n,$$

where

$$a_n = \frac{1}{n!} f^{(n)}(\alpha) \qquad (n = 0, 1, 2, \ldots).$$

This series is called the *Taylor series* for f at α; the expansion is valid on a disk $|z-\alpha| < R$ $(0 < R \leqq \infty)$, where R is the radius of convergence of the power series. The series is absolutely convergent on this disk, and uniformly convergent on any compact subset of it. Term-by-term differentiation is valid at any point within the circle of convergence.

If $f(z)$ is known to be regular on a domain D containing the point α, then the Taylor series expansion is certainly valid

on any open disk, centre α, contained in D. Moreover, if $f(z)$ is represented by a power series in $z-\alpha$ in a neighbourhood of α, then this series is necessarily the Taylor series for f at α.

For the basic elementary functions the Taylor series are:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{(all } z\text{)},$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \qquad \text{(all } z\text{)},$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \qquad \text{(all } z\text{)},$$

$$\log (1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \quad (|z| < 1;\ \text{principal branch}),$$

$$(1+z)^{\alpha} = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1) \ldots (\alpha-n+1)}{n!} z^n \quad (|z| < 1;\ \text{principal branch}).$$

From these, others may be derived by the processes of formal addition and subtraction, Cauchy product and formal substitution.

If a function f is not necessarily regular at α, but is regular in an annulus

$$r < |z-\alpha| < R \qquad (0 \leqq r < R \leqq \infty),$$

then there is a uniquely determined expansion, called the *Laurent series* at α,

$$f(z) = \sum_{-\infty}^{+\infty} A_n (z-\alpha)^n,$$

valid in the annulus (and possibly in some larger annulus with centre α). This "general power series" is to be regarded as the sum of

$$\sum_{n=0}^{\infty} A_n (z-\alpha)^n,$$

a power series in $z-\alpha$ which converges in $|z-\alpha| < R$, and

$$\sum_{n=1}^{\infty} A_{-n} \left(\frac{1}{z-\alpha}\right)^n,$$

a power-series in $1/(z-\alpha)$ which converges in $|z-\alpha| > r$.

An important special case occurs when $r = 0$ so that $f(z)$ is regular in the punctured disk $0 < |z-\alpha| < R$. There are then three possible types of behaviour of f at α, characterized by the form of the Laurent series.

(i) If $A_n = 0$ for all $n < 0$, then, by defining $f(\alpha) = A_0$, f becomes a *regular* function on the whole disk $|z-\alpha| < R$, and the given expansion is the Taylor series for f at α. For example, if

$$f(z) = \frac{\sin z}{z} \qquad (z \neq 0),$$

the Laurent series for $f(z)$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!},$$

so that f becomes a regular function on the whole plane, if we define $f(0) = 1$.

(ii) If, for some integer $m > 0$, $A_n = 0$ for all $n < -m$, $f(z)$ is said to have a *pole* at α; the pole has order m, if $A_{-m} \neq 0$. The finite series

$$\frac{A_{-m}}{(z-\alpha)^m} + \ldots + \frac{A_{-1}}{(z-\alpha)}$$

is called the *principal part* of the Laurent series. In this case, $g(z) = (z-\alpha)^m f(z)$ is regular at α, if $g(\alpha)$ is defined to be A_{-m}.

(iii) If the series of negative powers of $z-\alpha$ does not terminate, $f(z)$ is said to have an *essential singularity* at α.

In cases (ii) and (iii), the coefficient A_{-1} of $1/(z-\alpha)$ is called the *residue* of $f(z)$ at α.

A function whose only singularities on a domain D are poles is said to be *meromorphic* on D.

§ 0.8. Zeros of Analytic Functions

Suppose that $f(z)$ is regular at $z = \alpha$. If

$$f(z) = (z-\alpha)^r g(z), \quad g(\alpha) \neq 0, \quad r \geqq 1,$$

for some function $g(z)$ regular at α, we say that $f(z)$ has a *zero* of order r at $z = \alpha$. Equivalent conditions are that

$$f(\alpha) = f'(\alpha) = \ldots = f^{(r-1)}(\alpha) = 0, \; f^{(r)}(\alpha) \neq 0,$$

and that the first non-zero term of the Taylor series for f at α is $a_r(z-\alpha)^r$.

Let γ be a simple closed curve

$$z = z(t) \qquad (a \leqq t \leqq b)$$

and suppose that $f(z)$ is regular in the interior of γ and is continuous and never zero on γ. Then $f(z(t))$ is continuous and non-zero for $a \leqq t \leqq b$, and so for each t we may choose $\theta(t) = \arg f(z(t))$ so that $\theta(t)$ is a continuous function of t.

The "variation of $\arg f(z)$ around γ" is then defined to be

$$\theta(b)-\theta(a)$$

and may be denoted by

$$\Delta_\gamma \arg f(z).$$

Using the theory of complex integration, it can be shown that, if γ is positively oriented, the number of zeros of $f(z)$ inside γ (each counted according to its multiplicity) is given by

$$N = \frac{1}{2\pi} \Delta_\gamma \arg f(z).$$

This result is known as the *principle of the argument.*

From this result it is easy to deduce *Rouché's Theorem,* namely that, if $f(z)$ and $g(z)$ are regular inside and on γ and satisfy

$$|g(z)| < |f(z)| \text{ for } z \in \gamma,$$

then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside γ.

§ 0.9. Conformal Mapping

If f is a complex function, the relation

$$w = f(z),$$

where

$$w = u+iv, \; z = x+iy,$$

defines a mapping from the z-plane, with coordinates (x, y), into the w-plane, with coordinates (u, v).

Suppose that $f(z)$ is regular on a domain D, and that the curve

$$\gamma : z = z(t) \qquad (a \leqq t \leqq b)$$

lies in D. Then the map of γ in the w-plane is given by

$$w = w(t) = f(z(t)) \qquad (a \leqq t \leqq b),$$

which, since f is continuous on D, is also a curve Γ. If γ has a tangent at $z_0 = z(t_0)$, then, since

$$w'(t_0) = f'(z(t_0))z'(t_0) = f'(z_0)z'(t_0),$$

Γ has a tangent at the corresponding point $w_0 = f(z_0)$, provided that $f'(z_0) \neq 0$. Moreover, we have

$$\arg w'(t_0) = \arg f'(z_0) + \arg z'(t_0),$$

so that the angle between the (directed) tangents to γ at z_0 and to Γ at w_0 is $\arg f'(z_0)$, which is independent of γ.

We deduce that if γ_1, γ_2 are curves intersecting at z_0, and if $f'(z_0) \neq 0$, then their maps Γ_1, Γ_2 intersect at $w_0 = f(z_0)$ and form the same angle (in sense as well as magnitude). For this reason, the mapping by $w = f(z)$ is said to be *conformal* at points where $f'(z) \neq 0$.

Since also

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \lim_{z \to z_0} \frac{|w - w_0|}{|z - z_0|} = |f'(z_0)|,$$

any small line-segment at z_0 is expanded or contracted by a factor $|f'(z_0)|$, independently of its direction.

Combining the above results, we see that, if $f'(z_0) \neq 0$, a small figure at z_0 is mapped into an approximately similar figure at $w_0 = f(z_0)$, obtained by rotation through the angle $\arg f'(z_0)$ and by magnification by the factor $|f'(z_0)|$.

§ 1. Complex Numbers

1.1 Show that the number $(z_1+z_2)/(1+z_1z_2)$ is real, if $|z_1| = 1$, $|z_2| = 1$ and $z_1z_2 \neq -1$.

1.2 If $|a| = |b|$, then $|a+b|^2+|a-b|^2 = 4|a|^2$.

1.3 Prove the identity

$$|z_1+z_2|+|z_1-z_2| \equiv |z_1+\sqrt{z_1^2-z_2^2}|+|z_1-\sqrt{z_1^2-z_2^2}|.$$

HINT. Show that the squares of both sides of this identity are equal.

1.4 Prove the identity

$$|a|+|b| \equiv |\tfrac{1}{2}(a+b)-\sqrt{ab}|+|\tfrac{1}{2}(a+b)+\sqrt{ab}|. \qquad (1)$$

SOLUTION. The square of the right-hand side of the identity (1) is

$$|\tfrac{1}{2}(a+b)-\sqrt{ab}|^2+|\tfrac{1}{2}(a+b)+\sqrt{ab}|^2+2|\tfrac{1}{4}(a+b)^2-ab|.$$

Since $|z|^2 = z\bar{z}$, we obtain

$$\left(\frac{a+b}{2}-\sqrt{ab}\right)\left(\frac{\bar{a}+\bar{b}}{2}-\overline{\sqrt{ab}}\right)$$

$$+\left(\frac{a+b}{2}+\sqrt{ab}\right)\left(\frac{\bar{a}+\bar{b}}{2}+\overline{\sqrt{ab}}\right)+\tfrac{1}{2}|a-b|^2$$

$$= \tfrac{1}{2}[(a+b)(\bar{a}+\bar{b})+(a-b)(\bar{a}-\bar{b})]+2|ab|$$

$$= a\bar{a}+b\bar{b}+2|ab| = \{|a|+|b|\}^2.$$

We have proved that the squares of both sides of (1) are equal. Now it is easy to see that (1) is true.

REMARK. The identity (1) may be proved by putting $A = \sqrt{a}$ and $B = \sqrt{b}$ in the known identity

$$2\{|A|^2+|B|^2\} \equiv |A+B|^2+|A-B|^2.$$

1.5 1° Verify the identity

$$|a|^2+|b|^2+|c|^2+|a+b+c|^2 \equiv |a+b|^2+|b+c|^2+|c+a|^2.$$

2° Can this relation be generalized?

ANSWER. 2°:

$$\sum_{\nu=1}^{n}|a_\nu|^2+|\sum_{\nu=1}^{n}a_\nu|^2 = \sum_{1\leq\mu<\nu\leq n}|a_\mu+a_\nu|^2.$$

1.6 Prove the identity

$$|1-\bar{a}b|^2-|a-b|^2 \equiv (1+|ab|)^2-(|a|+|b|)^2.$$

HINT. Use the relation $|z|^2 = z\bar{z}$.

1.7 Prove the identities:

$$1^\circ \quad \left|b+\frac{1}{2}\right|^2+i\left|b+\frac{i}{2}\right|^2-(1+i)|b|^2-\frac{1}{4}(1+i) \equiv b,$$

$$2^\circ \quad |a+b|/|1+\bar{b}/a| \equiv a,$$

where $a > 0$, $b \neq -a$.

1.8 Determine the real and imaginary parts of the functions

1° $[a \exp it+b \exp i\theta]/[a \exp it-b \exp i\theta]$, where a, b, t, θ are real,

2° $1/(1-\exp z)$,

3° $1/\sin z$.

1.9 If $|a| = 1$ or $|b| = 1$, then $|(a-b)/(1-\bar{a}b)| = 1$. When is this relation true, if $|a| = |b| = 1$?

1.10 If $|z| = a > 0$, then $|kz-a^2| = a\,|z-k|$, k real.

1.11 If $x+iy = (u+iv)^n$, n an integer, then $x^2+y^2 = (u^2+v^2)^n$.

1.12 Express x^2+iy^2 as a function of z and $\bar{z}$.

1.13 Check the identity

$$(1+\cos a+i \sin a)^n \equiv 2^n \cos^n \frac{a}{2}\left(\cos\frac{na}{2}+i \sin\frac{na}{2}\right),$$

where a is real and n is a natural number.

1.14 Show that

$$\sqrt{a+ib} = \pm\left\{\sqrt{\frac{\sqrt{(a^2+b^2)}+a}{2}}+i \operatorname{sgn} b \sqrt{\frac{\sqrt{(a^2+b^2)}-a}{2}}\right\},$$

where a and b are real and $\operatorname{sgn} b = 1$ $(b \geqq 0)$, $\operatorname{sgn} b = -1\,(b < 0)$.

Examples:

$$\sqrt{(3+4i)} = \pm(2+i),\ \sqrt{(-7+24i)} = \pm(3+4i),\ \sqrt{i} = \pm\frac{1+i}{\sqrt{2}}.$$

1.15 Prove that $(1+i)^{4k}$ and $(1+i)^{4k+2}$ are real and purely imaginary, respectively.

SOLUTION. Since

$$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4},$$

we have

$$(1+i)^{4k} = \left(\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\right)^{4k} = 2^{2k}\,\mathrm{cis}\,k\pi = (-1)^k\,2^{2k}$$

and

$$\begin{aligned}(1+i)^{4k+2} &= (-1)^k\,2^{2k}\left(\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{4}\right)^2 = (-1)^k\,2^{2k+1}\,\mathrm{cis}\,\frac{\pi}{2}\\ &= (-1)^k\,2^{2k+1}\,i,\end{aligned}$$

as was to be proved.

1.16 Prove that $\operatorname{Re} z > 0$, $\operatorname{Re} a > 0$ implies $\left|\dfrac{a-z}{\bar{a}+z}\right| < 1$.

SOLUTION. Since

$$\begin{aligned}|a-z|^2-|\bar{a}+z|^2 &\equiv (a-z)(\bar{a}-\bar{z})-(\bar{a}+z)(a+\bar{z})\\ &\equiv -(a\bar{z}+\bar{a}z+az+\bar{a}\bar{z}) \equiv -(a+\bar{a})(z+\bar{z}) = -4\operatorname{Re} a\cdot\operatorname{Re} z,\end{aligned}$$

it follows that

$$|a-z|^2-|\bar{a}+z|^2 < 0 \text{ implies } |a-z|^2 < |\bar{a}+z|^2 \text{ implies } \left|\frac{a-z}{\bar{a}+z}\right|^2 < 1.$$

Is the converse true?

1.17 The moduli of the complex numbers a, b, c, d are the four successive terms of a geometric progression and their arguments are the four successive terms of an arithmetic progression. Determine b and c, if $a = \sqrt{2}$ and $d = 4i$.

ANSWER. The problem has the three solutions

$$\begin{aligned}&1^\circ \quad b = 2\,\mathrm{cis}\,\frac{\pi}{6}, && c = 2\sqrt{2}\,\mathrm{cis}\,\frac{\pi}{3},\\ &2^\circ \quad b = 2\,\mathrm{cis}\,\frac{5\pi}{6}, && c = 2\sqrt{2}\,\mathrm{cis}\,\frac{10\pi}{6},\\ &3^\circ \quad b = 2\,\mathrm{cis}\left(-\frac{\pi}{2}\right), && c = 2\sqrt{2}\,\mathrm{cis}\,(-\pi).\end{aligned}$$

1.18 Given

$$z_0 = x_0+iy_0 = r_0e^{i\theta_0}, \quad z_1 = x_1+iy_1 = r_1e^{i\theta_1}$$

and the quadratic equation

$$z^2-z_0z+z_1 = 0$$

with the roots a and b, determine the conditions to be satisfied by x_0, y_0, x_1, y_1 or r_0, θ_0, r_1, θ_1, in order that

1° $\arg a = \arg b$, 2° $\arg a-\arg b = \lambda$, 3° $|a|/|b| = \lambda$.

1.19 Let

$$f_1(z)=(z+|z|)/2,\ f_2(z) = f_1\{f_1(z)\}, \ldots, f_n(z) = f_1\{f_{n-1}(z)\}.$$

Evaluate $\lim_{n\to+\infty} f_n(i)$.

ANSWER. $2/\pi$.

REMARK. In general,

$$\lim_{n\to+\infty} f_n(z) = \frac{\operatorname{Im} z}{\arg z}, \text{ if } \arg z \neq 0,\ -\pi < \arg z \leqq +\pi,$$
$$= z \quad , \text{ if } \arg z = 0.$$

1.20 If x, y, θ are real and

$$\frac{3}{2+\cos\theta+i\sin\theta} = x+iy,$$

then $x^2+y^2 = 4x-3$.

1.21 Show that

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n \equiv \frac{1+i\tan n\theta}{1-i\tan n\theta},$$

where n is a natural number and $\theta \neq (2k+1)\pi/2$, $k = 0, \pm1, \pm2, \ldots$.

SOLUTION.

$$\frac{1+i\tan\theta}{1-i\tan\theta} = \frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta} = (\cos\theta+i\sin\theta)^2,$$

$$\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = (\cos\theta+i\sin\theta)^{2n} = \cos 2n\theta+i\sin 2n\theta,$$

$$\frac{1+i\tan n\theta}{1-i\tan n\theta} = (\cos n\theta+i\sin n\theta)^2 = \cos 2n\theta+i\sin 2n\theta,$$

and hence follows the result.

1.22 Prove the identity

$$(e^{it}+e^{i\theta})^n+(e^{-it}+e^{-i\theta})^n \equiv 2^{n+1}\cos^n\frac{\theta-t}{2}\cos n\frac{\theta+t}{2},$$

where t and θ are real and n is an integer.

1.23 Let $a \neq d$, $b \neq d$, $c \neq d$. Show that, if two of the three numbers $(a-b)/(c-d)$, $(b-c)/(a-d)$, $(c-a)/(b-d)$ are purely imaginary, then the third number is also purely imaginary.

SOLUTION. If we put $a = a_1+ia_2$, $b = b_1+ib_2$, $c = c_1+ic_2$, $d = d_1+id_2$, then

$$p^2\,\mathrm{Re}\left(\frac{a-b}{c-d}\right) = (a_1-b_1)(c_1-d_1)+(a_2-b_2)(c_2-d_2),$$

$$q^2\,\mathrm{Re}\left(\frac{b-c}{a-d}\right) = (b_1-c_1)(a_1-d_1)+(b_2-c_2)(a_2-d_2),$$

$$r^2\,\mathrm{Re}\left(\frac{c-a}{b-d}\right) = (c_1-a_1)(b_1-d_1)+(c_2-a_2)(b_2-d_2),$$

where $p^2 = |c-d|^2$, $q^2 = |a-d|^2$, $r^2 = |b-d|^2$. Adding these equations, we obtain

$$p^2\,\mathrm{Re}\left(\frac{a-b}{c-d}\right)+q^2\,\mathrm{Re}\left(\frac{b-c}{a-d}\right)+r^2\,\mathrm{Re}\left(\frac{c-a}{b-d}\right) = 0.$$

Hence we conclude that the assertion is true.

1.24 Prove that

$$|A|^2+|B|^2+|C|^2 = 3\{|x|^2+|y|^2+|z|^2\},$$

where

$$A = x+y+z, \quad B = x+\varepsilon y+\varepsilon^2 z, \quad C = x+\varepsilon^2 y+\varepsilon z,$$

x, y, z are complex and ε is a root of the equation $\varepsilon^2+\varepsilon+1 = 0$.

GENERALIZATION. Given the complex numbers $x_1, x_2, \ldots, x_n$, prove that

$$\sum_{k=0}^{n-1} |A_k|^2 = n \sum_{k=0}^{n-1} |x_{k+1}|^2,$$

where

$$A_k = x_1+x_2\,\varepsilon^k+x_3\varepsilon^{2k}+ \ldots +x_n\varepsilon^{(n-1)k}, \quad k = 0,\ 1,\ 2, \ldots, n-1,$$

$$\varepsilon = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}.$$

1.25 Check the relation

$$a \operatorname{Im}(\bar{b}c)+b \operatorname{Im}(\bar{c}a)+c \operatorname{Im}(\bar{a}b) = 0.$$

1.26 If $|a|=|b|=|c| = r > 0$, then $|(ab+bc+ca)/(a+b+c)|=r$.

HINT. Use the relation $|a|^2 = a\bar{a}$.

GENERALIZATION. H. L. Shug (The American Mathematical Monthly, **40**, 1933, p. 180) has given the following generalization: If $a_1, a_2, \ldots, a_n$ are complex numbers and

$$|a_1|=|a_2|= \ldots =|a_n|= r > 0,$$

then

$$|(\sum a_1 a_2 \ldots a_s)/(\sum a_1 a_2 \ldots a_{n-s})|= r^{2s-n} \quad (s < n), \qquad (1)$$

where the expressions in the numerator and the denominator are the elementary symmetric functions of $a_1, a_2, \ldots, a_n$ of order s and $n-s$, respectively.

We shall now quote Shug's proof: Firstly, note that, since $\binom{n}{s} = \binom{n}{n-s}$, the expressions

$$\sum a_1 a_2 \ldots a_s \text{ and } \sum a_1 a_2 \ldots a_{n-s}$$

have the same number of terms. Since $|a_k| = r$, we may put

$$a_k = r \exp(i\theta_k), \qquad k = 1,\ 2, \ldots, n.$$

Hence we obtain

$$\left|\frac{\sum a_1 a_2 \ldots a_s}{\sum a_1 a_2 \ldots a_{n-s}}\right| = r^{2s-n}\left|\frac{\sum \exp\,(i\sum_{k=1}^{s}\theta_k)}{\sum \exp\,(i\sum_{k=1}^{n-s}\theta_k)}\right|. \tag{2}$$

Evidently

$$\exp\,(i\sum_{k=1}^{s}\theta_k) = \exp\,(i\sum_{k=1}^{n}\theta_k)\exp\,(-i\sum_{k=s+1}^{n}\theta_k),$$

$$\sum \exp\,(i\sum_{k=1}^{s}\theta_k) = \exp\,(i\sum_{k=1}^{n}\theta_k)\sum \exp\,(-i\sum_{k=1}^{n-s}\theta_k),$$

and the expression (2) becomes

$$\left|\frac{\sum a_1 a_2 \ldots a_s}{\sum a_1 a_2 \ldots a_{n-s}}\right| = r^{2s-n}\,|\exp\,(i\sum_{k=1}^{n}\theta_k)|\left|\frac{\sum \exp\,(-i\sum_{k=1}^{n-s}\theta_k)}{\sum \exp\,(i\sum_{k=1}^{n-s}\theta_k)}\right|.$$

Finally, since $|\bar{z}| = |z|$ and $|\exp\,(i\theta)| = 1$, this expression agrees with (1).

1.27 Determine the minimum of the function

$$\max\,(|1+z|,\ |1+z^2|)$$

in the z-plane.

SOLUTION. The minimum may be written in the form

$$\begin{aligned}\min\max\,(|1+z|,\ |1+z^2|) &= \sqrt{\min\max\,(|1+z|^2,\ |1+z^2|^2)}\\ &= \sqrt{\min f(z)},\end{aligned} \tag{1}$$

where $f(z) = \max\,(|1+z|^2,\ |1+z^2|^2)$. If we put $z = \rho\exp\,(i\theta)$, we have

$$|1+z|^2 = |1+\rho\cos\theta+i\rho\sin\theta|^2 = 1+2\rho\cos\theta+\rho^2,$$
$$|1+z^2|^2 = |1+\rho^2\cos 2\theta+i\rho^2\sin 2\theta|^2 = 1+2\rho^2\cos 2\theta+\rho^4.$$

The expressions $|1+z|^2$ and $|1+z^2|^2$ are equal only for the points of the curve Γ, the equation of which is

$$1+2\rho^2\cos 2\theta+\rho^4 = 1+2\rho\cos\theta+\rho^2,$$

i.e.,

$$\rho^3 = 2\cos\theta+(1-2\cos 2\theta)\rho. \tag{2}$$

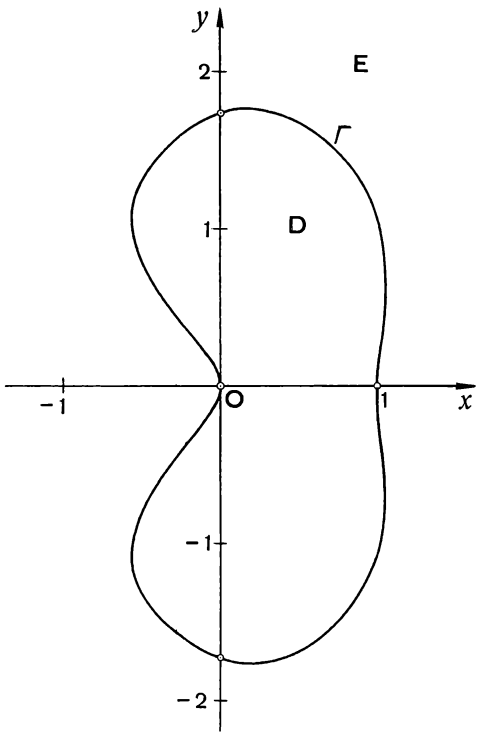

Fig. 2

The curve of Fig. 2 divides the z-plane into two parts: the region D enclosed by Γ and the region E outside Γ. Thus, we have

$$\begin{aligned} |1+z| &> |1+z^2| \text{ in } D, \\ |1+z| &= |1+z^2| \text{ on } \Gamma, \\ |1+z| &< |1+z^2| \text{ in } E. \end{aligned}$$

Therefore we may write

$$\begin{aligned} f(z) &= |1+z|^2 = 1+2\rho\cos\theta+\rho^2, && z \in D+\Gamma, \\ &= |1+z^2|^2 = 1+2\rho^2\cos 2\theta+\rho^4, && z \in E+\Gamma. \end{aligned}$$

The function $f(z) = \varphi(\rho, \theta)$ is differentiable in the regions D and E, but not on Γ. The function $f(z) = \varphi(\rho, \theta)$ assumes minimum values at points for which $d\varphi = 0$ or on Γ.

The condition $d\varphi = 0$ in D is equivalent to the equations

$$\frac{\partial\varphi}{\partial\rho} = 2\cos\theta + 2\rho = 0, \quad \frac{\partial\varphi}{\partial\theta} = -2\rho\sin\theta = 0.$$

Since these equations have no solutions in D, the function $\varphi(\rho, \theta)$ has no minima in D.

The condition $d\varphi = 0$ in E is equivalent to the equations

$$\frac{\partial\varphi}{\partial\rho} = 4\rho\cos 2\theta + 4\rho^3 = 0, \qquad \frac{\partial\varphi}{\partial\theta} = -4\rho^2\sin 2\theta = 0.$$

These equations have no solutions in E; hence the function $\varphi(\rho, \theta)$ has no minima in E.

Finally, since $f(\infty) = \infty$, we conclude that the function $f(z)$ can have a minimum on the curve Γ only. Thus, we have to determine the minimum of the function

$$F(\rho, \theta) = 1 + 2\rho\cos\theta + \rho^2,$$

where the variables ρ and θ satisfy the relation (2).

Using Lagrange multipliers, we start from the function

$$\Phi(\rho, \theta, \lambda) = 1 + 2\rho\cos\theta + \rho^2 + \lambda(\rho^4 + 2\rho^2\cos 2\theta - \rho^2 - 2\rho\cos\theta)$$

and find

$$\frac{\partial\Phi}{\partial\rho} = 2\cos\theta + 2\rho + \lambda(4\rho^3 + 4\rho\cos 2\theta - 2\rho - 2\cos\theta) = 0,$$

$$\frac{\partial\Phi}{\partial\theta} = -2\rho\sin\theta + \lambda(-4\rho^2\sin 2\theta + 2\rho\sin\theta) = 0,$$

$$\frac{\partial\Phi}{\partial\lambda} = \rho^4 + 2\rho^2\cos 2\theta - \rho^2 - 2\rho\cos\theta = 0.$$

Hence we obtain for ρ and θ:

1° $\rho = 0$, θ an arbitrary real number,

2° $\rho = 1$, $\theta = 0$,

3° $\rho^2 = \frac{1}{2}(3 \pm \sqrt{5})$, $\cos\theta = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)$.

By inspection, we conclude that

$$\min f(z) = 3-\sqrt{5},$$

and therefore, by (1), that

$$\min \max (|1+z|,\ |1+z^2|) = \sqrt{3-\sqrt{5}} = \frac{\sqrt{5}-1}{\sqrt{2}}.$$

GENERALIZATION. 1° Check the relation

$$\min \max (|1+z|,\ |z^2-z|,\ |1+z^2|) = \sqrt{3-\sqrt{5}}.$$

2° Determine

$$\min \max \{|f_n(z)-f_\nu(z)|;\ n \neq \nu;\ n,\ \nu = 1,\ 2, \ldots, m\},$$

where $f_1(z),\ f_2(z), \ldots, f_m(z)$ are polynomials.

REMARK. This problem is formulated in the journal: Matematikai Lapok, **8**, 1957, p. 290.

1.28 Prove the identity

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \varepsilon & \varepsilon^2 & & \varepsilon^{n-1} \\ 1 & \varepsilon^2 & \varepsilon^4 & & \varepsilon^{2(n-1)} \\ \vdots & & & & \\ 1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & & \varepsilon^{(n-1)(n-1)} \end{vmatrix} = \sqrt{n^n} \exp\left\{\frac{\pi}{4} i(n-1)(3n-2)\right\},$$

where $\varepsilon = \exp(2\pi i/n)$.

1.29 Prove that

$$A = \begin{vmatrix} 1 & a & \bar{a} \\ 1 & b & \bar{b} \\ 1 & c & \bar{c} \end{vmatrix}$$

is purely imaginary or zero.

SOLUTION. Since

$$\bar{A} = \begin{vmatrix} 1 & \bar{a} & a \\ 1 & \bar{b} & b \\ 1 & \bar{c} & c \end{vmatrix},$$

it follows that $\bar{A} = -A$, i.e., $A+\bar{A} = 0$; hence A is purely imaginary or zero.

1.30 Given n complex numbers z_1 and $z_k = (1/z_{k-1})^2$, $k = 2, 3, \ldots, n$, determine

1° z_k, $k = 2, 3, \ldots, n$, if $z_1 = 1+i$,

2° $\arg z_1$, so that the numbers z_k, $k = N, N+1, \ldots, n$, $(2 \leqq N \leqq n)$ are real,

3° z_1, so that $z_3 = z_1$.

Also show

4° that the points z_k, $k = 1, 2, \ldots, n$ lie on the logarithmic spiral $r = e^{a\theta}$ (r, θ polar coordinates). Is this curve determined when the point z_1 is given?

1.31 Determine a complex number a so that the function

$$f(z) = i\frac{ae^{iz}+1}{ae^{iz}-1}$$

satisfies the relation $f(\bar{z}) = \overline{f(z)}$.

§ 2. Zeros of Polynomials

2.1 Show that the fourth roots of i are

$$\pm\tfrac{1}{2}(\sqrt{2+\sqrt{2}}+i\sqrt{2-\sqrt{2}}),\ \pm\tfrac{1}{2}(-\sqrt{2-\sqrt{2}}+i\sqrt{2+\sqrt{2}}).$$

2.2 Determine the zeros of the polynomial

$$\operatorname{Re}\{(1+ix)^n\}-\operatorname{Im}\{(1+ix)^n\}$$

in x, where n is a natural number.

REMARK. All its zeros are real.

2.3 Let there be given points $A(a)$, $B(b)$ in the complex plane. If P is a point corresponding to a root of the equation $(z-a)^4+(z-b)^4 = 0$, then $|\overrightarrow{PA}| = |\overrightarrow{PB}|$. Generalize this result.

2.4 Evaluate

$$\cos\left(\frac{2}{5}\pi\right),\ \sin\left(\frac{2}{5}\pi\right),\ \sin\left(\frac{1}{5}\pi\right),\ \cos\left(\frac{1}{5}\pi\right).$$

HINT. Starting from the equation

$$z^5-1=0, \tag{1}$$

we obtain

$$z_k=\exp\left(\frac{2}{5}k\pi i\right), \qquad k=0,\ 1,\ 2,\ 3,\ 4.$$

The roots of (1) may also be found by solving the equations $z-1=0$ and $z^4+z^3+z^2+z+1=0$, which give

$$\frac{\sqrt{5}-1}{4}\pm\frac{i}{4}(10+2\sqrt{5})^{1/2}, \quad -\frac{\sqrt{5}+1}{4}\pm\frac{i}{4}(10-2\sqrt{5})^{1/2}.$$

Comparison of the two results yields the required expressions.

2.5 Express the function

$$ax^2+2bxy+cy^2+2dx+2ey+f \\ +i(Ax^2+2Bxy+Cy^2+2Dx+2Ey+F),$$

where $a,\ b,\ldots,A,\ B,\ldots$ are real parameters, in terms of z and $\bar{z}$. When is this function independent of $\bar{z}$?

2.6 Solve the equation $(z+1)^n=\exp(2nai)$, where a is real and n is a natural number. Evaluate

$$\prod_{k=0}^{n-1} z_k \quad \text{and} \quad \prod_{k=0}^{n-1}\sin\left(a+k\frac{\pi}{n}\right),$$

where the z_k, $k=0,\ 1,\ 2,\ldots,n-1$, are the roots of the given equation.

2.7 Check the identity

$$|z_1|+|z_2|=\frac{\sqrt{2}}{|a|}\sqrt{|b^2|+|ac|+|b^2-ac|}, \tag{1}$$

where z_1 and z_2 are the roots of the equation $az^2+2bz+c=0$.

SOLUTION. Letting $u+v=z_1$ and $u-v=z_2$, where

$$z_{1,2}=\frac{1}{a}\left[-b\pm(b^2-ac)^{1/2}\right],$$

the identity (see **1.4**)

$$|u+v|+|u-v| = |u+(u^2-v^2)^{1/2}|+|u-(u^2-v^2)^{1/2}|$$

becomes

$$|z_1|+|z_2| = \frac{1}{|a|}\,[|b+(ac)^{1/2}|+|b-(ac)^{1/2}|].$$

Squaring both sides we find

$$\begin{aligned}(|z_1|+|z_2|)^2 &= \frac{1}{|a|^2}\,[(b+\sqrt{ac})(\bar{b}+\overline{\sqrt{ac}}) \\ &\quad + (b-\sqrt{ac})(\bar{b}-\overline{\sqrt{ac}})+2|b^2-ac|] \\ &= \frac{2}{|a|^2}\,[|b|^2+|ac|+|b^2-ac|],\end{aligned}$$

and hence (1).

2.8 Determine the value of z if $|z/(z+1)| = 1$ and $z/\bar{z} = i$.

2.9 What condition must be satisfied by the real constants a and b so that the equation

$$\left(\frac{1+iz}{1-iz}\right)^n = a+ib, \tag{1}$$

n a natural number, has real and only real roots?

SOLUTION. If the equation (1) has a real root $z = x$, then $|(1+ix)/(1-ix)| = 1$ and also

$$a^2+b^2 = 1. \tag{2}$$

Thus, this relation represents the necessary condition. Now we shall show that (2) is also the sufficient condition. In fact, by (2), we may write $a+ib = \exp(i\theta)$, where θ is a real number. It follows then from (1) that

$$\frac{1+iz}{1-iz} = \exp\left(i\,\frac{\theta+2k\pi}{n}\right), \qquad k = 0,\ 1,\ 2,\ \ldots,\ n-1.$$

Hence

$$z_k = \tan\left(\frac{\theta+2k\pi}{2n}\right), \qquad k = 0,\ 1,\ 2,\ \ldots,\ n-1,$$

i.e., we obtain only real roots.

2.10 Determine all the roots of the equation $(z+a)^n = z^n$. If a is a real number, then all the roots lie on a straight line parallel to the imaginary axis.

ANSWER. $z_k = -(a/2)\{1+i \operatorname{cotan}(k\pi/n)\}$, $k = 1, 2, \ldots, n-1$.

2.11 Solve the equations

1° $az+b\bar{z}+c = 0$, $\quad 2^\circ$ $az\bar{z}+bz+c\bar{z}+d = 0$.

2.12 Solve the following equation for θ:

$$\prod_{k=1}^{n} (\cos k\theta + i \sin k\theta) = 1.$$

ANSWER. $\theta = 4m\pi/\{n(n+1)\}$, $m = 0, \pm 1, \pm 2, \ldots$.

2.13 Given the equation $az^2+bz+c = 0$, let z_1 be one of its roots. If $|z_1| = 1$, then

$$|\bar{a}b-\bar{b}c| = |a\bar{a}-c\bar{c}|. \tag{1}$$

SOLUTION. Since z_1 is a root of the given equation, one has $az_1^2+bz_1+c = 0$. Multiplying the last equation by $\bar{z}_1$ and using the fact that $z_1\bar{z}_1 = 1$, we obtain

$$az_1+b+c\bar{z}_1 = 0,$$

and hence

$$\bar{a}\bar{z}_1+\bar{b}+\bar{c}z_1 = 0.$$

Eliminating $\bar{z}_1$, we find

$$z_1 = -(\bar{a}b-\bar{b}c)/(a\bar{a}-c\bar{c}) \text{ and } |z_1| = |\bar{a}b-\bar{b}c|/|a\bar{a}-c\bar{c}|.$$

Finally, since $|z_1| = 1$, the above assertion follows.

REMARK. Examine the converse proposition: If (1) is satisfied, then the equation $az^2+bz+c = 0$ has a root z_1 for which $|z_1| = 1$.

2.14 Solve the equation

$$\sqrt[3]{z^2-2} = \sqrt{2-z^3}.$$

2.15 Determine all the roots of the equation $\bar{z} = z^{n-1}$, where $n > 2$ is a natural number.

SOLUTION. With $z = \rho e^{i\theta}$ the equation becomes

$$\rho e^{-i\theta} = \rho^{n-1} e^{(n-1)i\theta}.$$

Hence

$$\rho(\rho^{n-2}-1) = 0, \quad \theta = 2k\pi/n, \qquad k = 0, \pm 1, \pm 2, \ldots$$

and the roots of the equation are found to be

$$z = 0 \text{ and } z_k = \exp(2k\pi i/n), \qquad k = 0, 1, 2, \ldots, n-1.$$

REMARK. If $n = 0$, the equation is satisfied by every z for which $|z| = 1$. If $n = 1$, the only root is $z = 1$. If $n = 2$, any real z is a root.

2.16 Solve the equation $(\bar{z})^{\nu} = az^n$, where ν and n are natural numbers.

2.17 Determine a such that the roots of the equation

$$z^2-4z+a = 0$$

lie on a straight line with slope unity.

ANSWER. $\operatorname{Re} a = 4$, $\operatorname{Im} a \leqq 0$.

2.18 Determine the necessary and sufficient conditions so that the equation $z^2+2az+b$ will have
1° a real root,
2° an imaginary root,
3° two conjugate complex roots.

2.19 Solve the equation

$$z^4+8(1+2i)z^2-33+56i = 0.$$

ANSWER.

$$z_{1,2} = \pm(1-2i), \quad z_{3,4} = \pm(2-3i).$$

2.20 Prove that the equation $z^4+az+b = 0$, $a > 0$, $b > 0$, has one and only one root in the first quadrant of the complex plane and that the argument of this root is greater than $\pi/4$.

SOLUTION. Consider a contour C formed by
1° the line segment $z = x$ $(0 \leqq x \leqq R)$,
2° the arc $z = R e^{i\theta}$ $(0 \leqq \theta \leqq \pi/2)$,
3° the line segment $z = iy$ $(0 \leqq y \leqq R)$.

Denote by Δ_1, Δ_2, Δ_3 the variations of the argument of $f(z) = z^4+az+b$, when the point z describes in turn the component parts of C. Then the total variation along the contour C is

$$\Delta_C \arg f(z) = \Delta_1+\Delta_2+\Delta_3.$$

Since $f(z)$ is positive on the positive part of the x-axis, it follows that $\Delta_1 = 0$. Since $f(z) = z^4+o(z^4)$, we may conclude that $\Delta_2 \to 2\pi$ $(R \to +\infty)$. Finally, for $z = iy$, we obtain

$$f(z) = f(iy) = y^4+b+iay,$$

whence we conclude that $\Delta_3 \to 0$ when $R \to +\infty$.

Therefore it follows from (1) that

$$\lim_{R\to+\infty} \Delta_C \arg f(z) = 2\pi.$$

Thus the function $f(z)$ has one and only one zero in the first quadrant.

If z_1 is this zero of $f(z)$, then $x_1 > 0$, $y_1 > 0$ and $f(z_1) = 0$. Hence we have $\operatorname{Im} f(z_1) = 4x_1y_1(x_1^2-y_1^2)+ay_1 = 0$ and $x_1^2-y_1^2 = -a/4x_1$. Thus

$$y_1 > x_1,$$

and we conclude that $\arg z_1 > \pi/4$.

2.21 Show that, if a is real, the equation $z^5+z+a = 0$ has one imaginary root in each quadrant.

2.22 If all the zeros of the polynomial $P_n(z)$ lie on the same side of the straight line L, then all the zeros of the derivative $P_n'(z)$ lie in this region.

PROOF. By a suitably chosen transformation $z = z_0+w \exp i\theta$ we can map L into the imaginary axis of the w-plane, and all the zeros of $P_n(z)$ into the half-plane $\operatorname{Re} w < 0$. Then we have

$$P_n(z) = P_n(z_0+w \exp i\theta) = Q_n(w)$$

as well as

$$\mathrm{d}Q_n/\mathrm{d}w = e^{i\theta}\mathrm{d}P_n/\mathrm{d}z.$$

Thus, this transformation maps the zeros of $P_n(z)$ and $P_n'(z)$ into the zeros of $Q_n(w)$ and $Q_n'(w)$, respectively.

Let

$$Q_n(w) = a_0(w-w_1)^{k_1} \ldots (w-w_p)^{k_p},$$
$$a_0 \neq 0,\ k_1+k_2+\ldots+k_p = n,\ \operatorname{Re} w_\nu < 0.$$

The polynomial $Q_n'(w)$ has $p-1$ zeros which are not zeros of $Q_n(w)$. One has to show that these zeros lie on the left-hand side of the imaginary axis of the w-plane.

Let us assume that there exists a zero w of $Q_n'(w)$ (but not of $Q_n(w)$), so that $\operatorname{Re} w = u \geqq 0$. Then

$$\frac{Q'_n(w)}{Q_n(w)} = \frac{k_1}{w-w_1} + \frac{k_2}{w-w_2} + \ldots + \frac{k_p}{w-w_p} = 0,$$

and also

$$\frac{k_1[(u-u_1)-i(v-v_1)]}{|w-w_1|^2} + \frac{k_2[(u-u_2)-i(v-v_2)]}{|w-w_2|^2} + \ldots + \frac{k_p[(u-u_p)-i(v-v_p)]}{|w-w_p|^2} = 0.$$

Hence it follows that

$$\sum_{\nu=1}^{p} \frac{k_\nu(u-u_\nu)}{|w-w_\nu|^2} = 0,$$

which is impossible, since the terms in this sum are positive numbers.

Thus, our assumption is false and for every zero w of $Q_n'(w)$ one has $\operatorname{Re} w < 0$.

2.23 Prove Lucas' Theorem:

If the zeros of the polynomial $P_n(z)$ lie inside the circle C, then the zeros of the derivative $P_n'(z)$ lie also inside C.

HINT. Use the result of **2.22** with L tangent to C.

§ 3. Geometry of the Straight Line

3.1 Consider two points A and B of the complex plane corresponding to the complex numbers a and b. Determine

the complex number corresponding to the point B' obtained by rotating the vector $\overrightarrow{AB}$ about A by an angle θ.

3.2 Let A and B be the points corresponding to the complex numbers a and b, respectively. If $(a+b)^2/(ab)$ is real, then the points A, B and the origin of the complex plane are collinear or represent the vertices of an isosceles triangle.

3.3 Let there be given the complex numbers $a = 1+i\lambda$ and $b = 1-i/\lambda$, where λ is a real parameter. For what values of λ does the point a lie nearer to the origin than the point b?

3.4 The points z_1, z_2, z_3 are collinear, if and only if $(z_3-z_1)/(z_2-z_1) = a$, where a is real.

3.5 Show that the points $a-b$ and b lie symmetrically with respect to the point $a/2$.

3.6 Prove that the equation of the straight line through the points z_1 and $z_2 \neq z_1$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

3.7 Show that the equation of the straight line through the points z_1 and $z_2 \neq z_1$ is

$$\operatorname{Im}\left(\frac{z-z_1}{z_2-z_1}\right) = 0,$$

and that

$$\operatorname{Im}\left(\frac{z-z_1}{z_2-z_1}\right) > 0$$

defines the shaded region in Fig. 3.

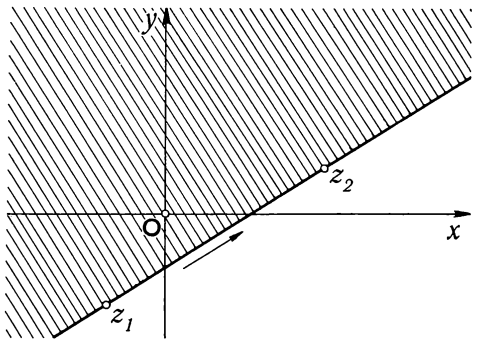

Fig. 3

3.8 Let the points A and B correspond to z_1 and z_2, where $z_1 z_2 \neq 0$.

1°: Prove that the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are orthogonal, if and only if

$$z_1\bar{z}_2+\bar{z}_1 z_2 = 0.$$

2°: Find the conditions under which the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are collinear.

ANSWER:

1°: The given relation yields step by step

$$z_1/\bar{z}_1 = -z_2/\bar{z}_2 \Leftrightarrow 2 \arg z_1 = 2 \arg z_2+(2k+1)\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

$$\Leftrightarrow \arg z_1-\arg z_2 = (2k+1)\pi/2.$$

2°: $\overrightarrow{OA} \parallel \overrightarrow{OB}$, if and only if

$$z_1\bar{z}_2 = \bar{z}_1 z_2.$$

3.9 Show that

$$z_1^2+z_2^2+z_3^2 = z_2z_3+z_3z_1+z_1z_2, \tag{1}$$

if and only if z_1, z_2, z_3 are the vertices of an equilateral triangle.

3.10 Prove that the roots of the equation

$$(az+b)^3 = 1, \qquad a \neq 0$$

represent the vertices of an equilateral triangle the length of the sides of which is $\sqrt{3}|a|$.

3.11 Prove that the points z_1, z_2, z_3 are collinear, if and only if

$$\lambda z_1+\mu z_2+\nu z_3 = 0, \quad \lambda+\mu+\nu = 0 \tag{1}$$

for some real λ, μ, ν, not all zero.

SOLUTION. The relations (1) are equivalent to

$$\lambda x_1+\mu x_2+\nu x_3 = 0, \ \lambda y_1+\mu y_2+\nu y_3 = 0, \ \lambda+\mu+\nu = 0.$$

These equations are consistent if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

i.e., if and only if the points (x_1, y_1), (x_2, y_2), (x_3, y_3) are collinear. (Cf. **3.6**).

3.12 Prove Pompeiu's theorem:

If ABC is an equilateral triangle and I is a point in the plane of this triangle, then $\overline{AI}$, $\overline{BI}$ and $\overline{CI}$ form a triangle.

PROOF. We shall quote Pompeiu's proof.

Let z_1, z_2, z_3 and z_0 correspond to the points A, B, C, I, respectively.

Subtracting the identities

$$z_1(z_2-z_3)+z_2(z_3-z_1)+z_3(z_1-z_2) \equiv 0,$$
$$z_0(z_2-z_3)+z_0(z_3-z_1)+z_0(z_1-z_2) \equiv 0,$$

we obtain

$$(z_0-z_1)(z_2-z_3)+(z_0-z_2)(z_3-z_1)+(z_0-z_3)(z_1-z_2) \equiv 0.$$

Since

$$|z_2-z_3| = |z_3-z_1| = |z_1-z_2|,$$

one finds from the last relation

$$|z_0-z_1| \leqq |z_0-z_2|+|z_0-z_3|,$$
$$|z_0-z_1| \geqq |z_0-z_2|-|z_0-z_3|,$$

i.e.,

$$|\overline{IB}-\overline{IC}| \leqq \overline{IA} \leqq \overline{IB}+\overline{IC},$$

and, similarly,

$$|\overline{IC}-\overline{IA}| \leqq \overline{IB} \leqq \overline{IC}+\overline{IA},$$
$$|\overline{IA}-\overline{IB}| \leqq \overline{IC} \leqq \overline{IA}+\overline{IB}.$$

These relations show that $\overline{IA}$, $\overline{IB}$ and $\overline{IC}$ can be the sides of a triangle (including the limiting case when it reduces to a line segment).

3.13 Let z_1 and z_3 be two opposite vertices of a square. Determine its other vertices.

ANSWER. $\frac{1}{2}(z_1+z_3)\pm\frac{1}{2}i(z_1-z_3)$.

3.14 If the points A, B, C, D are coplanar, then

$$\overline{AD}\cdot\overline{BC} \leqq \overline{BD}\cdot\overline{CA}+\overline{CD}\cdot\overline{AB}.$$

PROOF. Let the complex numbers z_1, z_2, z_3 and z_4 correspond to the points A, B, C, D, respectively. Since

$$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2) \equiv 0,$$

one has

$$|(z_1-z_4)(z_2-z_3)| \equiv |(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)|,$$

and hence

$$|z_1-z_4|\,|z_2-z_3| \leqq |z_2-z_4|\,|z_3-z_1|+|z_3-z_4|\,|z_1-z_2|,$$

which proves the statement.

3.15 If z_ν $(\nu = 1, 2, 3, 4)$ are the successive vertices of a quadrangle inscribed in a circle, then the following relations hold:

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)} > 0, \tag{1}$$

$$|z_1-z_3|\,|z_2-z_4| = |z_1-z_2|\,|z_3-z_4|+|z_1-z_4|\,|z_2-z_3|. \tag{2}$$

Explain the geometrical meaning of (2).

SOLUTION. If a is the center and r the radius of the circle passing through the points z_ν ($\nu = 1, 2, 3, 4$), then we may write

$$z_\nu = a + r\, e^{\theta_\nu i},$$

where

$$\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + 2\pi. \tag{3}$$

Therefore we obtain

$$\begin{aligned}
\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_2-z_3)} &= \frac{(e^{\theta_1 i}-e^{\theta_2 i})(e^{\theta_3 i}-e^{\theta_4 i})}{(e^{\theta_1 i}-e^{\theta_4 i})(e^{\theta_2 i}-e^{\theta_3 i})} \\
&= \frac{\left(e^{\frac{\theta_1-\theta_2}{2} i} - e^{-\frac{\theta_1-\theta_2}{2} i}\right)\left(e^{\frac{\theta_3-\theta_4}{2} i} - e^{-\frac{\theta_3-\theta_4}{2} i}\right)}{\left(e^{\frac{\theta_1-\theta_4}{2} i} - e^{-\frac{\theta_1-\theta_4}{2} i}\right)\left(e^{\frac{\theta_2-\theta_3}{2} i} - e^{-\frac{\theta_2-\theta_3}{2} i}\right)} \\
&= \frac{\sin\frac{\theta_2-\theta_1}{2}\sin\frac{\theta_4-\theta_3}{2}}{\sin\frac{\theta_4-\theta_1}{2}\sin\frac{\theta_3-\theta_2}{2}} > 0,
\end{aligned}$$

since, by (3), $0 < |(\theta_i-\theta_k)/2| < \pi$, $i, k = 1, 2, 3, 4$. This proves (1).

If $a/b > 0$, then $|a/b+1| = |a/b|+1$ and also $|a+b| = |a|+|b|$. Hence we obtain from (1)

$$\begin{aligned}
&|(z_1-z_2)(z_3-z_4)|+|(z_1-z_4)(z_2-z_3)| \\
&= |(z_1-z_2)(z_3-z_4)+(z_1-z_4)(z_2-z_3)| = |-z_1z_4-z_2z_3+z_1z_2+z_4z_3| \\
&= |(z_1-z_3)(z_2-z_4)|,
\end{aligned}$$

i.e., the relation (2).

The relation (2) represents Ptolemy's theorem: *The product of the diagonals of a quadrangle inscribed in a circle is equal to the sum of the products of the two pairs of the opposite sides.*

3.16 1° Let z_k, $k = 1, 2, 3, 4$, correspond to the points M_k. Prove that the relation

$$z_2 - z_1 = \pm i(z_4 - z_3) \tag{1}$$

is satisfied, if and only if

$$\overline{M_1M_2} = \overline{M_3M_4} \text{ and } M_1M_2 \perp M_3M_4. \tag{2}$$

2° Let α and β correspond to the points A and B in the complex plane. Determine the number m corresponding to the vertex M of the right-angled, isosceles triangle AMB, where AMB is the positive orientation of the triangle.

3° Let there be given a quadrilateral $ABCD$ and four isosceles right-angled triangles AMB, BNC, CPD and DQA, where M, N, P, Q are the vertices corresponding to the right angles. If the circumferences of these triangles are oriented in the same direction, then $\overline{MP} = \overline{NQ}$ and $MP \perp NQ$.

SOLUTION. 1° By (1),

$$|z_2 - z_1| = |z_4 - z_3|,$$

i.e., $\overline{M_1M_2} = \overline{M_3M_4}$. From (1) we also deduce that

$$\arg(z_2 - z_1) = \arg[\pm i(z_4 - z_3)] = \arg \pm i + \arg(z_4 - z_3),$$

and hence

$$\arg(z_2 - z_1) = \arg(z_4 - z_3)\ (\pm i)\ \frac{\pi}{2},$$

i.e., that $M_1M_2 \perp M_3M_4$. Thus, (2) follows from (1).

It is clear that $\overline{M_1M_2} = |z_2 - z_1|$ and $\overline{M_3M_4} = |z_4 - z_3|$. It follows from the first of the relations (2) that $|z_2 - z_1| = |z_4 - z_3|$. Denoting by r the moduli of the numbers $z_2 - z_1$ and $z_4 - z_3$, write $z_2 - z_1 = re^{\theta i}$, $z_4 - z_3 = re^{\theta' i}$. Hence

$$z_2 - z_1 = e^{(\theta' - \theta)i}(z_4 - z_3). \tag{3}$$

The second of the relations (2) shows that

$$\theta' - \theta = \pm \frac{\pi}{2} + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

and (3) becomes $z_2 - z_1 = \pm i(z_4 - z_3)$. Thus, (2) implies (1).

2° Since $\overline{MA} = \overline{MB}$ and $MA \perp MB$, we obtain by 1°, the relation $\alpha - m = \pm i(\beta - m)$. Since the triangle AMB is oriented in the positive direction, we have, finally,

$$m = \frac{\alpha - i\beta}{1-i}. \tag{4}$$

3° Let α, β, γ, δ, m, n, p, q be the complex numbers corresponding to the points A, B, C, D, M, N, P, Q, respectively. By (4), we obtain, assuming all orientations to be positive,

$$m = \frac{\alpha - i\beta}{1-i},\ n = \frac{\beta - i\gamma}{1-i},\ p = \frac{\gamma - i\delta}{1-i},\ q = \frac{\delta - i\alpha}{1-i}. \tag{5}$$

In order to prove that $\overline{MP} = \overline{NQ}$ and $MP \perp NQ$, it is sufficient to show that $n-q = i(m-p)$. Substituting m, n, p, q from (5), we find

$$\frac{\beta - i\gamma}{1-i} - \frac{\delta - i\alpha}{1-i} = i\left(\frac{\alpha - i\beta}{1-i} - \frac{\gamma - i\delta}{1-i}\right),$$
$$\beta - \delta + i(\alpha - \gamma) = i[\alpha - \gamma + i(\delta - \beta)],$$
$$\beta - \delta + i(\alpha - \gamma) = \beta - \delta + i(\alpha - \gamma),$$

and hence $n-q = i(m-p)$, as was to be proved.

3.17 Determine the coefficients a, b, c, d such that the points of the z-plane, corresponding to the roots of the equation

$$z^5 + az^4 + bz^3 + cz^2 + dz = 0, \tag{1}$$

represent the vertices of a regular pentagon with center at the point $z = 1$.

SOLUTION. The points corresponding to the roots of the equations

$$z^5 = -1 \tag{2}$$

are the vertices of a regular pentagon with center at the origin and with a vertex at the point $z = -1$. Letting $z = w-1$, we obtain from (2) the equation

$$(w-1)^5 = -1. \tag{3}$$

Its roots are the vertices of a regular pentagon with center at the point $w = 1$ and with a vertex at the point $w = 0$.

The equation (1) is equivalent to the equation (3), if and only if $a = -5$, $b = 10$, $c = -10$, $d = 5$.

3.18 Given two consecutive vertices z_0 and z_1 of a regular polygon of n sides, determine its other vertices.

SOLUTION. Denote by B_0 and B_1 the vertices of this polygon at the points z_0 and z_1. Assume that its center A is at the point a. Then

$$\sphericalangle (\overrightarrow{AB}_0, \overrightarrow{AB}_1) = \theta = \pm \frac{2\pi}{n}. \tag{1}$$

Thus, if $\overrightarrow{AB}_0$ turns about the point A through the angle θ, we obtain the vector $\overrightarrow{AB}_1$. Using the result of **3.1**, we find

$$z_1 = a + (z_0 - a)e^{i\theta}.$$

Hence the centre of the polygon is given by

$$a = z_0 + \frac{z_1 - z_0}{1 - e^{i\theta}}. \tag{2}$$

If AB_0 turns about the point A through the angle $k\theta$, we obtain the vector AB_k, k an integer, $0 \leqq k \leqq n-1$. The point B_k is the $(k+1)$-th vertex of the polygon, traversed in the direction from B_0 to B_1. If B_k is at z_k, then, again by **3.1**, we find

$$z_k = a + (z_0 - a)e^{ik\theta}$$

and, by (2),

$$z_k = z_0 + (z_1 - z_0)\,\frac{1 - \exp ik\theta}{1 - \exp i\theta}.$$

Finally, by (1), we find for the z_k the two sets of values

$$z_k^{\pm} = z_0 + (z_1 - z_0)\,\frac{1 - \exp(\pm 2k\pi i/n)}{1 - \exp(\pm 2\pi i/n)}$$

$$= z_0 + (z_1 - z_0)\exp\{\pm(k-1)\pi i/n\}\,\frac{\sin(k\pi/n)}{\sin(\pi/n)},$$

since

$$\frac{1-\exp(\pm 2k\pi i/n)}{1-\exp(\pm 2\pi i/n)} = \exp\{\pm(k-1)\pi i/n\}\frac{\sin(k\pi/n)}{\sin(\pi/n)}.$$

Thus, we have obtained two distinct regular polygons with n sides having two common vertices B_0, B_1, as had to be expected.

3.19 The regular polygon $P_1, P_2, \ldots, P_n$ is inscribed in an unit circle C.

1° Evaluate $\sum_{k=1}^{n} \overline{PP_k}^2$, where P is a point on the circle C.

2° Prove the relation

$$\sum_{k=1}^{n} \overline{PP_k}^2 = n(1+\overline{OP}^2),$$

where O is the centre of C and P is a point in the plane of the given polygon.

SOLUTION. 1° Let the centre of the circle C be at the origin of the complex plane and the vertices P_k of the given polygon at the points z_k, $k = 1, 2, \ldots, n$, respectively. Assume that P_1 lies on the positive x-axis. If the point P corresponds to the number z, then

$$\overline{PP_k} = |z-z_k|,$$
$$\overline{PP_k}^2 = |z-z_k|^2 = (z-z_k)(\bar{z}-\bar{z}_k) = z\bar{z}+z_k\bar{z}_k-z\bar{z}_k-\bar{z}z_k.$$

Since the points z and z_k lie on the unit circle, one finds from this relation

$$\overline{PP_k}^2 = 2-z\bar{z}_k-\bar{z}z_k.$$

Hence

$$\sum_{k=1}^{n}(2-z\bar{z}_k-\bar{z}z_k) = 2n-z\sum_{k=1}^{n}\bar{z}_k-\bar{z}\sum_{k=1}^{n}z_k.$$

Since $z_k = \exp\{(k-1)(2\pi i)/n\}$, $k = 1, 2, \ldots, n$, we obtain

$$\sum_{k=1}^{n} z_k = 0, \quad \sum_{k=1}^{n} \bar{z}_k = 0,$$

and thus

$$\sum_{k=1}^{n} \overline{PP_k}^2 = 2n.$$

3.20 A regular polygon of n sides is inscribed in a circle with radius r.

1° Prove that the sum of the squares of all the sides and of all the diagonals of the given polygon is n^2r^2.

2° Prove that the sum of all the sides and all the diagonals is $nr \operatorname{cotan} (\pi/2n)$.

3° Prove that the product of all the sides and all the diagonals is $n^{n/2}\, r^{n(n-1)/2}$.

SOLUTION 1°. Let the centre of the circle be at the origin of the complex plane and let the vertex P_1 be at the point $z = r$. Then the vertices P_k, $k = 1, 2, \ldots, n$, are at the points

$$z_k = r \exp \{(k-1)2i\pi/n\}. \tag{1}$$

By **3.19**, one has

$$\sum_{k=1}^{n} \overline{P_1P_k}^2 = 2nr^2, \quad \sum_{k=1}^{n} \overline{P_2P_k}^2 = 2nr^2, \ldots, \quad \sum_{k=1}^{n} \overline{P_nP_k}^2 = 2nr^2.$$

In these sums every quantity $\overline{P_sP_k}$ appears twice. If we denote by S the sum of the squares of all the sides and of all the diagonals, then we find

$$S = \tfrac{1}{2}n \cdot 2nr^2 = n^2r^2. \tag{2}$$

The sum S may also be determined in the following manner. It is given by the $\binom{n}{2}$ terms

$$\begin{aligned} S = {} & |z_1-z_2|^2+|z_1-z_3|^2+ \ldots +|z_1-z_n|^2 \\ & +|z_2-z_3|^2+ \ldots +|z_2-z_n|^2 \\ & + \ldots \\ & +|z_{n-1}-z_n|^2. \end{aligned} \tag{3}$$

Since

$$(z_p-z_q)(\bar{z}_p-\bar{z}_q) = 2r^2-\bar{z}_pz_q-z_p\bar{z}_q,$$

we obtain by (1)

$$\sum_{p<q} (\bar{z}_p z_q + z_p \bar{z}_q)$$
$$= 2r^2 \left[\cos\frac{2\pi}{n} + \cos 2\cdot\frac{2\pi}{n} + \ldots + \cos(n-2)\frac{2\pi}{n} + \cos(n-1)\frac{2\pi}{n} \right.$$
$$+ \cos\frac{2\pi}{n} + \cos 2\cdot\frac{2\pi}{n} + \ldots + \cos(n-2)\frac{2\pi}{n}$$
$$+ \ldots$$
$$\left. + \cos\frac{2\pi}{n} \right] = 2r^2\sigma,$$

and hence

$$S = 2r^2 \binom{n}{2} - 2r^2\sigma.$$

Comparing this result with (2), we find

$$\sigma = \sum_{\nu=1}^{n-1} (n-\nu) \cos\left(\nu\cdot\frac{2\pi}{n}\right) = -\tfrac{1}{2}n. \tag{4}$$

REMARK 1: D. Djoković has obtained the result (4) in the following steps:

$$\sigma = \sum_{\nu=1}^{n-1} \nu \cos\left[(n-\nu)\frac{2\pi}{n}\right] = \sum_{\nu=1}^{n-1} \nu \cos\left(2\pi-\nu\frac{2\pi}{n}\right) = \sum_{\nu=1}^{n-1} \nu \cos\left(\nu\frac{2\pi}{n}\right)$$
$$= \frac{n}{2\pi}\left[\frac{\mathrm{d}}{\mathrm{d}x}\sum_{\nu=1}^{n-1} \sin\left(\nu\frac{2\pi}{n}x\right)\right]_{x=1}$$
$$= \frac{n}{2\pi}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{\sin\pi x}{\sin\left(\frac{\pi}{n}x\right)}\sin\left(\frac{n-1}{n}\pi x\right)\right\}\right]_{x=1} = -\frac{1}{2}n.$$

REMARK 2: V. Janekoski obtained (4) in the following manner: Let

$$G(\theta) = \sum_{\nu=0}^{n-1} (n-\nu)\cos\nu\theta + i\sum_{\nu=0}^{n-1} (n-\nu)\sin\nu\theta = n\sum_{\nu=0}^{n-1} e^{i\nu\theta} - \sum_{\nu=0}^{n-1} \nu e^{i\nu\theta}. \tag{1'}$$

Since

$$\sum_{\nu=0}^{n-1} \nu e^{i\nu\theta} = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{\nu=0}^{n-1} e^{i\nu\theta}, \quad \sum_{\nu=0}^{n-1} e^{i\nu\theta} = \frac{e^{in\theta}-1}{e^{i\theta}-1}, \ \theta \neq 2k\pi,\ k = 0, \pm 1, \pm 2, \ldots,$$

we obtain from (1′)

$$G(\theta) = \frac{n}{2}\left(1 + i\,\frac{\cos\dfrac{\theta}{2}}{\sin\dfrac{\theta}{2}}\right) + e^{i\theta}\,\frac{e^{in\theta}-1}{(e^{i\theta}-1)^2}. \tag{2′}$$

Setting in (1′) and (2′) $\theta = 2\pi(\lambda/n)$, where λ and n are integers, we find

$$G\left(\frac{2\pi}{n}\lambda\right) = \sum_{\nu=0}^{n-1}(n-\nu)\cos\frac{2\pi\nu\lambda}{n} + i\sum_{\nu=0}^{n-1}(n-\nu)\sin\frac{2\pi\nu\lambda}{n},$$

$$G\left(\frac{2\pi}{n}\lambda\right) = \frac{n}{2}\left(1 + i\,\operatorname{cotan}\frac{\pi\lambda}{n}\right),$$

and hence, by comparison,

$$\sum_{\nu=0}^{n-1}(n-\nu)\sin\frac{2\pi\nu\lambda}{n} = \frac{n}{2}\operatorname{cotan}\frac{\pi\lambda}{n}, \quad \sum_{\nu=0}^{n-1}(n-\nu)\cos\frac{2\pi\nu\lambda}{n} = \frac{n}{2}.$$

Finally, letting $\lambda = 1$, we have

$$\sum_{\nu=0}^{n-1}(n-\nu)\sin\frac{2\pi\nu}{n} = \frac{n}{2}\operatorname{cotan}\frac{\pi}{n}, \quad \sum_{\nu=0}^{n-1}(n-\nu)\cos\frac{2\pi\nu}{n} = \frac{n}{2}.$$

3.21 Consider a regular polygon $P_1P_2\ldots P_{2m}$, inscribed in a circle with radius r, and any point P. Show that

$$\sum_{k=1}^{m}\overline{PP}_{2k}^2 = \sum_{k=1}^{m}\overline{PP}_{2k-1}^2.$$

PROOF. Let the centre of the circle be at the origin and $z = re^{i\theta}$ denote the point P. If the vertex P_1 of the given polygon is at the point $z_1 = r$, the other vertices are given by $z_k = r\exp\{(k-1)i\pi/m\}$, $k = 1, 2, \ldots, 2m$. We must now prove that

$$\sum_{k=1}^{m}|z-z_{2k}|^2 = \sum_{k=1}^{m}|z-z_{2k-1}|^2. \tag{1}$$

Since

$$\sum_{k=1}^{m}\exp\{(2k-1)i\pi/m\} = 0, \quad \sum_{k=1}^{m}\exp\{(2k-2)i\pi/m\} = 0,$$

we obtain

$$\sum_{k=1}^{m} |z-z_{2k}|^2 = 2mr^2 - \bar{z}\sum_{k=1}^{m} z_{2k} - z\sum_{k=1}^{m} \bar{z}_{2k} = 2mr^2,$$

$$\sum_{k=1}^{m} |z-z_{2k-1}|^2 = 2mr^2 - \bar{z}\sum_{k=1}^{m} z_{2k-1} - z\sum_{k=1}^{m} \bar{z}_{2k-1} = 2mr^2,$$

and hence (1).

3.22 Given a regular polygon of n sides inscribed in a circle with radius r, show that the sum of the diagonals of this polygon is

$$nr \sin\frac{(n-3)\pi}{2n} \Big/ \left(\sin\frac{\pi}{2n}\right).$$

§ 4. Geometry of Conics

4.1 Show that the equation

$$z\bar{z} + \bar{A}z + A\bar{z} + B = 0, \tag{1}$$

where B is real and A complex, represents a circle in the complex plane.

SOLUTION. The circle with radius r and centre at the point α is represented by the equation $|z-\alpha| = r$. Then $(z-\alpha)(\bar{z}-\bar{\alpha}) = r^2$, and hence

$$z\bar{z} - \bar{\alpha}z - \alpha\bar{z} + \alpha\bar{\alpha} - r^2 = 0.$$

Comparing this equation with (1), we obtain

$$\alpha = -A, \quad r^2 = \alpha\bar{\alpha} - B = |\alpha|^2 - B.$$

If $|A|^2 > B$, then the circle is real. If $|A|^2 = B$, the equation (1) represents the point $z = -A$.

REMARK. This problem may be solved by setting $x = (z+\bar{z})/2$ and $y = (z-\bar{z})/2i$ in the general equation of the circle $x^2+y^2+px+qy+s = 0$.

4.2 Does the equation $|z-p| = 2|z|$ represent the circle with radius $(2/3)|p|$ and centre at the point $(-1/3)p$?

4.3 Show that the equation of a circle through three points z_k, $k = 1, 2, 3$ is represented by

$$\begin{vmatrix} z\bar{z} & z & \bar{z} & 1 \\ z_1\bar{z}_1 & z_1 & \bar{z}_1 & 1 \\ z_2\bar{z}_2 & z_2 & \bar{z}_2 & 1 \\ z_3\bar{z}_3 & z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0. \tag{1}$$

SOLUTION. Expanding the determinant (1), we obtain

$$D_1 z\bar{z}+D_2 z+D_3\bar{z}+D_4 = 0, \tag{2}$$

where

$$D_1 = \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}, \quad D_2 = -\begin{vmatrix} z_1\bar{z}_1 & \bar{z}_1 & 1 \\ z_2\bar{z}_2 & \bar{z}_2 & 1 \\ z_3\bar{z}_3 & \bar{z}_3 & 1 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} z_1\bar{z}_1 & z_1 & 1 \\ z_2\bar{z}_2 & z_2 & 1 \\ z_3\bar{z}_3 & z_3 & 1 \end{vmatrix}, \quad D_4 = -\begin{vmatrix} z_1\bar{z}_1 & z_1 & \bar{z}_1 \\ z_2\bar{z}_2 & z_2 & \bar{z}_2 \\ z_3\bar{z}_3 & z_3 & \bar{z}_3 \end{vmatrix}.$$

It is easily seen that D_1 and D_4 are purely imaginary and $D_2 = \bar{D}_3$. If we assume that $D_1 \neq 0$, i.e., that the points z_1, z_2 and z_3 are not collinear, then (2) may be written in the form

$$z\bar{z}+\bar{d}z+d\bar{z}+c = 0,$$

where $d = D_3/D_1$ and $c = D_4/D_1$. Since c is real, the last equation, by **4.1**, represents a circle.

4.4 Show that the equation of a circle with diameter P_1P_2 is $|2z-z_1-z_2| = |z_1-z_2|$, where z_1 and z_2 are the complex numbers corresponding to the points P_1 and P_2, respectively.

4.5 If α, β, γ are real and a, b, c complex, then the equation

$$z = x+iy = (a+2bt+ct^2)/(\alpha+2\beta t+\gamma t^2),$$

where t is a real variable, represents a conic section.

4.6 What does the equation

$$\sum_{k=1}^{n} a_k|z-z_k|^2 = b$$

represent in the complex plane, if a_k, $k = 1, 2, 3, \ldots, n$, and b are real?

ANSWER. The given equation defines a circle or a straight line if

$$\sum_{k=1}^{n} a_k \neq 0 \quad \text{or} \quad \sum_{k=1}^{n} a_k = 0,$$

respectively.

4.7 If the points z_1, z_2 and z_3 lie on the unit circle with centre at the origin and satisfy the relation

$$z_1+z_2+z_3 = 0, \tag{1}$$

then they represent the vertices of an equilateral triangle.

SOLUTION. By hypothesis, we have

$$|z_1| = 1, \quad |z_2| = 1, \quad |z_3| = 1. \tag{2}$$

The sides of the triangle with vertices at z_1, z_2 and z_3 are $|z_2-z_1|$, $|z_3-z_1|$ and $|z_3-z_2|$. Since $z_3 = -z_1-z_2$ and $z_1\bar{z}_1 = 1$, $z_2\bar{z}_2 = 1$, we obtain

$$|z_3-z_1|^2=|2z_1+z_2|^2=(2z_1+z_2)(2\bar{z}_1+\bar{z}_2)=5+2(z_1\bar{z}_2+\bar{z}_1z_2),$$
$$|z_3-z_2|^2=|2z_2+z_1|^2=(2z_2+z_1)(2\bar{z}_2+\bar{z}_1)=5+2(z_1\bar{z}_2+\bar{z}_1z_2).$$

Hence

$$|z_3-z_1|=|z_3-z_2|. \tag{3}$$

In a similar manner, the relations (1) and (2) lead to

$$|z_2-z_1|=|z_3-z_2|. \tag{4}$$

Hence $|z_2-z_1|=|z_3-z_1|=|z_3-z_2|$, i.e., the given triangle is equilateral.

4.8 Show that, if the points z_1, z_2, z_3, z_4 lie on the circle $|z| = 1$ and if

$$z_1+z_2+z_3+z_4 = 0, \tag{1}$$

then they represent the vertices of a rectangle.

4.9 Given a set of n concentric circles in the z-plane with centre at the origin. Let $A_\nu B_\nu$, $\nu = 1, 2, \ldots, n$, be arcs of these circles oriented in the same direction with central angle θ. Denote by C_ν, $\nu = 1, 2, \ldots, n$ the midpoints of the chords $B_\nu A_{\nu+1}$ $(A_{n+1} = A_1)$.

1° Determine θ such that the complex numbers, corresponding to the points C_ν, $\nu = 1, 2, \ldots, n$, are linearly dependent without regard to the choice of the points A_ν, $\nu = 1, 2, \ldots, n$.

2° Explain the geometric meaning of the linear dependence for $n = 3$ and $n = 4$.

SOLUTION. 1° Let z_ν, $z_\nu e^{\theta i}$, ζ_ν be the complex numbers corresponding to the points A_ν, B_ν, C_ν, respectively. Since C_ν is the midpoint of the chord $B_\nu A_{\nu+1}$, we have

$$\zeta_\nu = \tfrac{1}{2}(z_\nu e^{i\theta} + z_{\nu+1}), \qquad \nu = 1, 2, \ldots, n.$$

The condition 1° is satisfied if and only if

$$\begin{vmatrix} e^{i\theta} & 1 & 0 & & & 0 \\ 0 & e^{i\theta} & 1 & & & \cdot \\ & & e^{i\theta} & & & \cdot \\ & & & & & \\ & & & & & \\ & & & 1 & & \\ & & & e^{i\theta} & 1 & \\ 1 & & & & e^{i\theta} & \end{vmatrix} = 0.$$

Expanding this determinant with respect to the first row, we find the required relations

$$e^{in\theta} = (-1)^n, \quad \text{i.e.,} \quad \theta = \pi + 2k\pi/n, \quad k = 0, \pm 1, \pm 2, \ldots$$

Thus, for these values of θ, the ζ_ν, $\nu = 1, 2, \ldots, n$ satisfy the linear dependence relation

$$\begin{vmatrix} \zeta_1 & 1 & & & 0 \\ \zeta_2 & e^{i\theta} & 1 & & \\ \zeta_3 & 0 & e^{i\theta} & & \\ \vdots & & & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ \zeta_n & 0 & & & e^{i\theta} \end{vmatrix} = 0,$$

i.e.,

$$\zeta_n - \alpha\zeta_{n-1} + \alpha^2\zeta_{n-2} - \ldots + (-1)^{n-1}\alpha^{n-1}\zeta_1 = 0, \quad \alpha = e^{i\theta}. \tag{1}$$

2° For $n = 3$, the relation (1) becomes

$$\zeta_3 - \alpha\zeta_2 + \alpha^2\zeta_1 = 0, \quad \alpha = e^{i\pi/3}, \quad \frac{\zeta_3 - \zeta_1}{\zeta_2 - \zeta_1} = \alpha.$$

Hence we conclude that

$$|\overrightarrow{C_1C_3}| = |\overrightarrow{C_1C_2}|, \quad \sphericalangle\,(\overrightarrow{C_1C_2}, \overrightarrow{C_1C_3}) = \pi/3.$$

Thus the triangle $C_1C_2C_3$ is equilateral.

If $n = 4$, then

$$\zeta_4 - i\zeta_3 - \zeta_2 + i\zeta_1 = 0, \quad \frac{\zeta_4 - \zeta_2}{\zeta_3 - \zeta_1} = i.$$

Hence it follows that

$$|\overrightarrow{C_2C_4}| = |\overrightarrow{C_1C_3}|, \quad \sphericalangle\,(\overrightarrow{C_1C_3}, \overrightarrow{C_2C_4}) = \pi/2.$$

Therefore the diagonals of the quadrilateral $C_1C_2C_3C_4$ are equal and perpendicular.

REMARK. This is a generalization of a problem of F. Denk (Glasnik matematičko-fizički i astronomski, 11(2), 1956, p. 87) due to D. Djoković.

4.10 If the point z describes an ellipse E_1 with foci at $z_1 = 2$, $z_2 = -2$, the point $w = z^3 - 3z$ describes an ellipse E_2. Show that the ellipses E_1 and E_2 are confocal.

SOLUTION. The point z which describes the ellipse E_1 may be represented by the expression

$$z = a\cos t + ib\sin t, \quad a^2 - b^2 = 4, \quad 0 \leqq t < 2\pi. \tag{1}$$

Hence, the point w is given by

$$\begin{aligned} w = {} & a\cos t(a^2\cos^2 t - 3b^2\sin^2 t - 3) \\ & + ib\sin t(3a^2\cos^2 t - b^2\sin^2 t - 3), \end{aligned} \tag{2}$$

and, since $a^2-b^2 = 4$, we find

$$a^2\cos^2 t-3b^2\sin^2 t-3 = (a^2-3)(1-4\sin^2 t),$$
$$3a^2\cos^2 t-b^2\sin^2 t-3 = (b^2+3)(4\cos^2 t-1).$$

Thus, when z moves along E_1, $w = u+iv$ moves along

$$u = (a^3-3a)(1-4\sin^2 t)\cos t,$$
$$v = (b^3+3b)(4\cos^2 t-1)\sin t,\quad a^2-b^2 = 4,\quad 0 \leqq t < 2\pi,$$

i.e., along the ellipse

$$u = (a^3-3a)\cos 3t,\quad v = (b^3+3b)\sin 3t,$$
$$a^2-b^2 = 4,\quad 0 \leqq t < 2\pi.$$

Since $(a^3-3a)^2-(b^3+3b)^2 = 4$, it follows that the ellipses E_1 and E_2 are confocal. In fact, the point w describes the ellipse E_2 three times as the point z moves along the ellipse E_1.

§ 5. Functions and Loci

5.1 Determine the locus of the point z satisfying the relation $\operatorname{Im}(z+1/z-i\log z) = 0$.

SOLUTION. If we put $z = \rho\exp i\theta$, we obtain

$$z+\frac{1}{z}-i\log z = \left\{\left(\rho+\frac{1}{\rho}\right)\cos\theta+\theta+2k\pi\right\}$$
$$+i\left\{\left(\rho-\frac{1}{\rho}\right)\sin\theta-\log\rho\right\}.$$

Thus the equation of this locus is

$$\left(\rho-\frac{1}{\rho}\right)\sin\theta = \log\rho.$$

Trace this curve.

5.2 Determine the loci defined by the relations:

1° $z = \exp it$, 2° $z = t\exp i\pi/6$,
3° $z = t+\exp it$, 4° $z = a+b\exp it$
5° $z = a+bt+ct^2$,

where t is real.

5.3 What locus is represented by

$$\operatorname{Im}\left(\frac{z}{z-1}\right)^6 = 0? \tag{1}$$

GENERALIZATION. What locus is represented by

$$\operatorname{Im}\left(\frac{z-a}{z-b}\right)^n = 0, \tag{2}$$

where n is a natural number?

SOLUTION. Since

$$\frac{z}{z-1} = \frac{x+iy}{x-1+iy} = \frac{(x^2+y^2-x)-iy}{(x-1)^2+y^2},$$

it follows from (1) that

$$\operatorname{Im}\{(x^2+y^2-x)-iy\}^6 = 0$$

and

$$y(x^2+y^2-x)\{3(x^2+y^2-x)^4-10(x^2+y^2-x)^2y^2+3y^4\} = 0.$$

From the equation

$$3(x^2+y^2-x)^4-10(x^2+y^2-x)^2y^2+3y^4 = 0$$

we obtain

$$(x^2+y^2-x)^2 = 3y^2 \quad \text{or} \quad (x^2+y^2-x)^2 = \tfrac{1}{3}y^2.$$

Now, it is readily seen that the required locus consists of a straight line and five circles from which the point (1, 0) has been excluded:

$$y = 0,$$

$$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2,$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{\sqrt{3}}{2}\right)^2 = 1,$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{\sqrt{3}}{2}\right)^2 = 1,$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\frac{\sqrt{3}}{3}\right)^2 = \left(\frac{\sqrt{3}}{3}\right)^2,$$

$$\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\frac{\sqrt{3}}{3}\right)^2 = \left(\frac{\sqrt{3}}{3}\right)^2.$$

We shall now consider the general case. Since

$$\operatorname{Im}\left(\frac{z-a}{z-b}\right)^n = \frac{1}{2i}\left[\left(\frac{z-a}{z-b}\right)^n - \left(\frac{\bar{z}-\bar{a}}{\bar{z}-\bar{b}}\right)^n\right],$$

we obtain from (2)

$$\left(\frac{z-a}{z-b}\right)^n = \left(\frac{\bar{z}-\bar{a}}{\bar{z}-\bar{b}}\right)^n, \quad \text{i.e.,} \quad \left[\frac{(z-a)(\bar{z}-\bar{b})}{(z-b)(\bar{z}-\bar{a})}\right]^n = 1.$$

Hence

$$\frac{(z-a)(\bar{z}-\bar{b})}{(z-b)(\bar{z}-\bar{a})} = \alpha^k, \quad \alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \; k = 0, 1, 2, \ldots, n-1,$$

and

$$(1-\alpha^k)z\bar{z} + (\bar{a}\alpha^k - \bar{b})z + (b\alpha^k - a)\bar{z} + (a\bar{b} - \alpha^k\bar{a}b) = 0.$$

Multiplying this equation by $1+\bar{\alpha}^k$, we obtain

$$\left(-2i\sin\frac{2k\pi}{n}\right)z\bar{z} + (\bar{a}\alpha^k - \bar{b} + \bar{a} - \bar{b}\bar{\alpha}^k)z + (b\alpha^k - a + b - a\bar{\alpha}^k)\bar{z},$$
$$+ (a\bar{b} - \alpha^k\bar{a}b + a\bar{b}\bar{\alpha}^k - \bar{a}b) = 0,$$

i.e.,

$$\left(2\sin\frac{2k\pi}{n}\right)z\bar{z} + i[(\bar{a}-\bar{b}) + (\bar{a}\alpha^k - \bar{b}\bar{\alpha}^k)]z - i[(a-b) + (a\bar{\alpha}^k - b\alpha^k)]\bar{z}$$
$$+ i[(a\bar{b} - \bar{a}b) + (a\bar{b}\bar{\alpha}^k - \alpha^k\bar{a}b)] = 0. \tag{3}$$

The coefficients of z and $\bar{z}$ are conjugate complex, the coefficient of $z\bar{z}$ and the constant term are real. Thus, for $k = 1, 2, 3, \ldots, n-1$, the equation (3) represents circles passing through the points a and b.

If $k = 0$, the equation (3) reduces to

$$(\bar{a}-\bar{b})z - (a-b)\bar{z} + (a\bar{b} - \bar{a}b) = 0,$$

the straight line through the points a and b.

Hence it follows that the required locus consists of a straight line and $n-1$ circles from which the point b has been excluded.

REMARK. D. Djoković has given the following solution. If we write $z-a = \rho e^{i\theta}$, $z-b = re^{i\varphi}$, the relation (2) becomes $\operatorname{Im}\left((\rho^n/r^n)e^{in(\theta-\varphi)}\right)=0$. Hence we obtain $\theta-\varphi = k\pi/n$, $k = 0, 1, 2, \ldots, 2n-1$.

On the basis of elementary geometry, we may now arrive at the following conclusions:

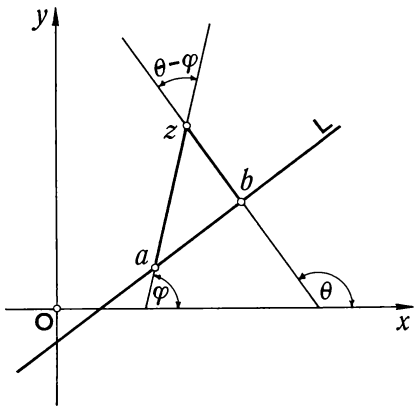

Fig. 4

If $k = 0$, then the point z describes both parts of the line L (see Fig. 4) outside the line segment ab. For $k = n$, we obtain the line segment ab.

If $k = \nu$, $0 < \nu < n$, the point z describes the arc of a circle lying on the left-hand side of the straight line L, if we move from a to b. For $k = n+\nu$, we obtain the remainder of the same circle.

Thus, the required locus consists of the straight line L and the $n-1$ circles passing through the points a and b. The point b does not belong to this locus.

5.4 Determine the locus

$$z = (t^2+8it+3)/(t^2-2it+3),$$

where t is a real parameter.

HINT. Evaluate $z\bar{z}$ and $z+\bar{z}$ and eliminate t between these expressions to obtain the circle

$$z\bar{z}+\tfrac{3}{2}(z+\bar{z})-4 = 0.$$

5.5 What locus is represented by $(a\bar{z}+\bar{a}z)^2 = 2(b\bar{z}+\bar{b}z)+c$, c real?

5.6 Trace in the complex plane the curve defined by the equation $|z-a| = \lambda|z-b|$, where $\lambda \geqq 0$.

REMARK. If $0<\lambda\neq 1$, $a\neq b$, one has a circle with radius $\lambda|(a-b)/(\lambda^2-1)|$ and centre at $(\lambda^2 b-a)/(\lambda^2-1)$.

5.7 Let

$$f(z)=\{(z-a)/(z-\bar{a})\}^2.$$

What locus is represented by $|f(z)| = c^2$, c real?

SOLUTION. We have

$$|f(z)|=|(z-a)/(z-\bar{a})|^2 = \frac{z-a}{z-\bar{a}}\cdot\frac{\bar{z}-\bar{a}}{\bar{z}-a} = \frac{z\bar{z}-a\bar{z}-\bar{a}z+a\bar{a}}{z\bar{z}-\bar{a}\bar{z}-az+a\bar{a}},$$

and hence

$$(c^2-1)z\bar{z}+(\bar{a}-ac^2)z+(a-\bar{a}c^2)\bar{z}+|a|^2(c^2-1) = 0. \qquad (1)$$

Since the coefficients c^2-1 and $|a|^2(c^2-1)$ are real and $\bar{a}-ac^2$ and $a-\bar{a}c^2$ are conjugate, this equation represents a circle.

If $c = \pm 1$, then (1) represents a straight line (degenerate circle).

5.8 Trace the locus of points z satisfying the relation $|\exp iz+1| = 1$.

5.9 Determine the locus of points z satisfying the relations
1° $\operatorname{Im}(\exp z) = 0$, 2° $\operatorname{Re}(\exp z) = 0$.

5.10 Show that

$$|\sin(x+iy)|=|\sin x+\sin iy|,\ |\sinh(x+iy)|=|\sinh x+\sinh iy|,$$

where x and y are real.

REMARK. R. M. Robinson (The American Mathematical Monthly, **64**, 1957, p. 83—85) has proved the following theorem:

If $f(z)$ is regular for $|z| < r$, and satisfies the relation $|f(x+iy)| = |f(x)+f(iy)|$, where x and y are real, then it has the form Az or $A\sin bz$ or $A\sinh bz$, where A is complex and b real.

E. Hille in the article "A Pythagorean functional equation" (Annals of Mathematics, **24**, 1922, p. 175—180) has obtained the same result for the functional equation $|f(x+iy)|^2 = |f(x)|^2+|f(iy)|^2$.

5.11 Determine the locus of points z satisfying the relation

$$\operatorname{Re}\{(1+i)z\} = 0.$$

5.12 Given the analytic function $w(z) = u(x, y) + e^{\theta i} v(x, y)$, where θ is real, show that the angle between the curves $u(x, y) = \text{const.}$, $v(x, y) = \text{const.}$ is θ.

5.13 Let M_1 and M_2 be the points corresponding to the roots z_1 and z_2 of the quadratic equation

$$z^2 - 2(\alpha + i\beta)z - 2 - 2i = 0, \tag{1}$$

where α and β are real.

1° Determine the locus of the point $P(\alpha, \beta)$, if the slope of the straight line passing through the points M_1 and M_2 is 1.

2° Determine the locus of the point $P(\alpha, \beta)$, if $\sphericalangle M_1 O M_2$ is a right angle, where O is the origin.

SOLUTION. 1° By hypothesis, we have

$$\tan \arg (z_1 - z_2) = 1$$

and

$$\arg (z_1 - z_2) = \frac{\pi}{4} + k\pi. \tag{2}$$

From (1), we obtain

$$z_1 = \alpha + i\beta + \sqrt{(\alpha + i\beta)^2 + 2 + 2i},$$
$$z_2 = \alpha + i\beta - \sqrt{(\alpha + i\beta)^2 + 2 + 2i},$$

and hence

$$z_1 - z_2 = 2\sqrt{(\alpha + i\beta)^2 + 2 + 2i}.$$

Thus,

$$\arg (z_1 - z_2) = \tfrac{1}{2} \operatorname{arc\,tan} 2 \frac{\alpha\beta + 1}{\alpha^2 - \beta^2 + 2}$$

and, by (2),

$$\tfrac{1}{2} \operatorname{arc\,tan} 2 \frac{\alpha\beta + 1}{\alpha^2 - \beta^2 + 2} = \frac{\pi}{4} + k\pi,$$

i.e.,

$$\alpha^2 - \beta^2 + 2 = 0, \qquad \alpha\beta + 1 > 0.$$

The required locus consists of two hyperbolic arcs, which are parts of the hyperbola $\alpha^2 - \beta^2 + 2 = 0$ ending at the points

$$\left(\sqrt{\sqrt{2} - 1}, -\sqrt{\sqrt{2} + 1}\right), \quad \left(-\sqrt{\sqrt{2} - 1}, \sqrt{\sqrt{2} + 1}\right).$$

For the end points, $\alpha^2-\beta^2+2=0$, $\alpha\beta+1=0$, and hence $z_1=z_2$. Thus the straight line passing through the points M_1 and M_2 is not determined.

2° If $\sphericalangle M_1OM_2=\pi/2$, then $z_2=kiz_1$, where k is real. Since $z_1+z_2=z_1(1+ki)$ and $z_1-z_2=z_1(1-ki)$, we conclude that

$$|z_1+z_2|=|z_1-z_2|.$$

Since

$$z_1+z_2=\alpha+i\beta,\ z_1-z_2=\sqrt{(\alpha+i\beta)^2+2+2i},$$

we obtain

$$\alpha^2-\beta^2+2\alpha\beta+2=0$$

which is again a hyperbola.

5.14 Let $f(z)$ be a complex polynomial in the variable $z=x+iy$ and $G(x, y)$, $H(x, y)$ two complex polynomials in the real variables x, y. Show that, if $f(z)=G(x, y)H(x, y)$, then $G(x, y)=g(z)$, $H(x, y)=h(z)$, where $g(z)$ and $h(z)$ are polynomials in z.

PROOF. Let $x=z_1-iz_2$, $y=z_2$, z_1, z_2 being two independent complex variables; then the identity $f(x+iy)\equiv G(x, y)$. $H(x, y)$ reduces to $f(z_1)\equiv G(z_1-iz_2,\ z_2)H(z_1-iz_2,\ z_2)$. Since the left-hand side of the last identity depends on z_1 only, the polynomials G and H do not depend on z_2 as was to be proved.

REMARK. This problem was proposed by L. Carlitz in the journal "Elemente der Mathematik", **13**, 1962, No. 5, p. 114.

5.15 Determine $\lim_{|z|\to+\infty}|(\exp z)/z^n)|$, n a natural number, if

$$1^\circ\quad -\frac{\pi}{2}<\arg z<\frac{\pi}{2},\quad 2^\circ\quad \frac{\pi}{2}<\arg z<\frac{3\pi}{2}.$$

5.16 Prove the identities

$$\cos z=\cos x\cosh y-i\sin x\sinh y,$$
$$\sin z=\sin x\cosh y+i\cos x\sinh y,$$
$$\tan z=\frac{\tan x(1-\tanh^2 y)}{1+\tan^2 x\tanh^2 y}+i\frac{\tanh y\,(1+\tan^2 x)}{1+\tan^2 x\tanh^2 y},$$

$$\cosh z = \cosh x \cos y + i \sinh x \sin y,$$
$$\sinh z = \sinh x \cos y + i \cosh x \sin y,$$
$$\tanh z = \frac{\tanh x(1+\tan^2 y)}{1+\tanh^2 x \tan^2 y} + i\frac{\tan y(1-\tanh^2 x)}{1+\tanh^2 x \tan^2 y}.$$

5.17 Prove the identities and inequalities

$$|\sin z| = \sqrt{\sin^2 x+\sinh^2 y}, \qquad |\cos z| = \sqrt{\cos^2 x+\sinh^2 y},$$
$$|\sinh z| = \sqrt{\sin^2 y+\sinh^2 x}, \qquad |\cosh z| = \sqrt{\cos^2 y+\sinh^2 x},$$
$$|\sinh y| \leqq |\sin z| \leqq \cosh y, \qquad |\sinh y| \leqq |\cos z| \leqq \cosh y,$$
$$|\sinh x| \leqq |\sinh z| \leqq \cosh x, \qquad |\sinh x| \leqq |\cosh z| \leqq \cosh x.$$

5.18 Prove

$$1^\circ\ \sinh (m\pi i) = 0, \quad 2^\circ\ \cosh (m\pi i) = \cos m\pi = (-1)^m,$$
$$3^\circ\ \tanh (m\pi i) = 0, \quad 4^\circ\ \sinh (x+m\pi i) = (-1)^m \sinh x,$$
$$5^\circ\ \cosh (x+m\pi i) = (-1)^m \cosh x,$$

where m is an integer.

5.19 Determine $\max |\sin z|$ in the domain $|x| \leqq 1$, $|y| \leqq 1$.

HINT. Start from the identity $|\sin z| = \sqrt{\sin^2 x+\sinh^2 y}$.

5.20 Determine the maximum and the minimum of $|z|$ when z satisfies the relation $|z+b/z| = a$, $a > 0$.

SOLUTION. With $z = \rho e^{i\theta}$, $b = Be^{i\beta}$, the relation $|z+b/z| = a$ takes the form

$$\left| \rho \cos\theta + \frac{B}{\rho}\cos(\beta-\theta) + i\left\{ \rho \sin\theta + \frac{B}{\rho}\sin(\beta-\theta) \right\} \right| = a,$$

that is

$$\rho^2 + \frac{B^2}{\rho^2} = a^2 - 2B\cos(\beta-2\theta). \tag{1}$$

Now, let

$$y = f(x) = x + \frac{B^2}{x}, \qquad x > 0, \tag{2}$$

$$y = a^2 - 2B\cos(\beta-2\theta), \tag{3}$$

where $x = \rho^2$, and determine the conditions under which the

variable x attains a maximum and a minimum.

The functions (2) and (3) correspond to the hyperbola and the straight line parallel to the x-axis, respectively, shown in Fig. 5.

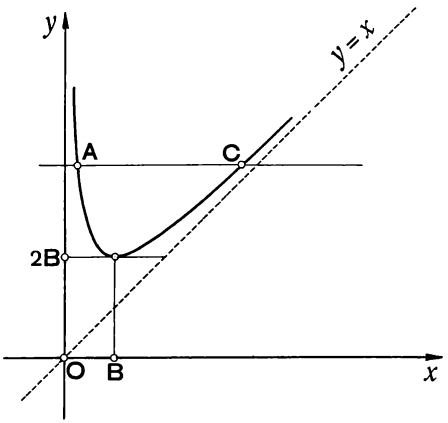

Fig. 5

Hence we can see that the extreme values of x are the abscissae of the points A and C when the straight line AC lies furthest away from the x-axis. This case occurs when $\cos(\beta-2\theta) = -1$, i.e., for

$$y = a^2+2B. \tag{4}$$

Since $x = \rho^2$, we obtain from (2) and (4) the equation

$$\rho^2 + \frac{B^2}{\rho^2} = a^2+2B,$$

for the determination of the extreme values of ρ.

The positive roots of this equation are

$$\rho = \frac{+\sqrt{a^2+4B}\pm a}{2}.$$

Thus

$$\min |z| = \tfrac{1}{2}(\sqrt{a^2+4|b|}-a),$$
$$\max |z| = \tfrac{1}{2}(\sqrt{a^2+4|b|}+a).$$

REMARK 1° The problem can be solved starting from the relation

$$|z| = \left|z+\frac{b}{z}-\frac{b}{z}\right| \leqq \left|z+\frac{b}{z}\right| + \frac{|b|}{|z|} = a + \frac{|b|}{|z|}.$$

REMARK 2° In the Russian journal Matematicheskoe Prosveshchenie (1960, No. 5, p. 265-266) this problem is solved for $b = 1$.

5.21 Evaluate $\lim_{z\to\infty} \tan z$ under the condition that

$$\arg z = \alpha \neq k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

SOLUTION. Writing $z = \rho \exp(i\theta)$, one has

$$\tan(\rho e^{i\theta}) = \frac{\sin(2\rho\cos\theta) + i\sinh(2\rho\sin\theta)}{\cosh(2\rho\sin\theta) + \cos(2\rho\cos\theta)}$$

$$= \tanh(2\rho\sin\theta) \cdot \frac{i + \{\sin(2\rho\cos\theta)\}/\{\sinh(2\rho\sin\theta)\}}{1 + \{\cos(2\rho\cos\theta)\}/\{\cosh(2\rho\sin\theta)\}}.$$

If $2k\pi < \theta < (2k+1)\pi$, k an integer, then

$$\tanh(2\rho\sin\theta) \to 1, \qquad \rho \to \infty.$$

If $(2k-1)\pi < \theta < 2k\pi$, then

$$\tanh(2\rho\sin\theta) \to -1, \qquad \rho \to \infty.$$

Finally, for $(2k-1)\pi < \theta < 2k\pi$ and $2k\pi < \theta < (2k+1)\pi$, we have

$$\frac{i + \{\sin(2\rho\cos\theta)\}/\{\sinh(2\rho\sin\theta)\}}{1 + \{\cos(2\rho\cos\theta)\}/\{\cosh(2\rho\sin\theta)\}} \to i, \qquad \rho \to \infty.$$

Thus, for $\operatorname{Im}(z) > 0$,

$$\tan z \to i \text{ as } z \to \infty,$$

for $\operatorname{Im}(z) < 0$,

$$\tan z \to -i \text{ as } z \to \infty.$$

§ 6. Inequalities and Regions *

6.1 Prove that

$$\left|\frac{z_1}{z_1 + z_2}\right| \leqq \frac{|z_1|}{||z_1| - |z_2||}, \quad |z_1| \neq |z_2|.$$

6.2 Interpret geometrically the relation $|1 - z|^2 \leqq 1 - |z|^2$.

HINT. This relation is equivalent to the relation

$$\left|z - \tfrac{1}{2}\right| \leqq \tfrac{1}{2}.$$

6.3 If $\operatorname{Re} z \geqq 0$, then

* See also: D. S. Mitrinović, Elementary Inequalities, Tutorial Text No. 1, Noordhoff, Groningen, 1964, pp. 138-143

$$|1+z| \geqq \frac{1+|z|}{\sqrt{2}}. \tag{1}$$

SOLUTION. Assume that there is a value of z so that

$$|1+z| < \frac{1+|z|}{\sqrt{2}}.$$

Then we have also

$$(1+z)(1+\bar{z}) < \frac{(1+|z|)^2}{2}$$

and

$$2(z+\bar{z}) < -(|z|-1)^2.$$

Hence it follows that

$$4\,\mathrm{Re}(z) < -(|z|-1)^2.$$

This relation contradicts the assumption $\mathrm{Re}\, z \geqq 0$, and hence (1) follows.

6.4 If $a > 0$, $b > 1$ and

$$\mathrm{Re}\, z \geqq \frac{(2-b^2)a}{2(b^2-1)},$$

then

$$|a+z| \geqq \frac{a+|z|}{b}.$$

6.5 If $x > 1$, then

$$\left|\frac{1}{z} - \frac{1}{2}\right| < \frac{1}{2}. \tag{1}$$

PROOF. I. Assume the assertion to be false. Then we have

$$\left|\frac{1}{z} - \frac{1}{2}\right| \geqq \frac{1}{2},$$

and also

$$\left(\frac{1}{z} - \frac{1}{2}\right)\left(\frac{1}{\bar{z}} - \frac{1}{2}\right) \geqq \frac{1}{4}.$$

Hence

$$\frac{1-x}{x^2+y^2} \geqq 0,$$

and, finally, $x \leqq 1$, which contradicts the condition $x > 1$. This proves the above proposition.

II. Erik Dux (Matematikai Lapok, **8**, fasc. 1—2, 1957, p. 148) has given the following proof:

From $x > 1$ it follows that $4-4x < 0$ and

$$4-4x+x^2+y^2 \equiv (2-x)^2+y^2 < x^2+y^2.$$

Hence

$$|(2-x)-iy| < |x+iy|, \text{ i.e., } |2-z| < |z|$$

and

$$\left|\frac{1}{z}-\frac{1}{2}\right| < \frac{1}{2}.$$

6.6 Prove that

$$\frac{|x|+|y|}{\sqrt{2}} \leqq |z|.$$

HINT. Assume that this relation is false. Equality holds, if and only if $|x|=|y|$.

6.7 Prove that

$$\left|\frac{|a|-|b|}{|c|+|d|}\right| \leqq \left|\frac{a+b}{c+d}\right| \leqq \left|\frac{|a|+|b|}{|c|-|d|}\right|.$$

6.8 If $|z_k| < 1$, $k = 1, 2, \ldots, n$, then

$$\left|\prod_{k=1}^{n}(1-z_k)\right| \geqq 1-\sum_{k=1}^{n}|z_k|.$$

6.9 Determine the regions of the complex plane defined by the relations

1° $|z-i| \leqq 1$, $|z-1| \leqq 1$,
2° $\pi/6 \leqq \arg(z-1-i) \leqq \pi/3$.

6.10 Determine the domain in the complex plane defined by the relations $|z-a| < r$ and $|\arg(z-a)| < \theta$ $(r > 0,\ 0 < \theta < \pi)$.

6.11 Determine the regions of the complex plane defined by the relations:

$$1^\circ\ |z| < 1 - \operatorname{Re} z, \qquad 2^\circ\ 0 < \arg\frac{z-i}{z+i} < \frac{\pi}{4}.$$

6.12 Let a_ν, $\nu = 1, 2, \ldots, n$ and z be any complex numbers and $c = (1/n)\sum_{\nu=1}^{n} a_\nu$.

Prove the following relations:

$$1^\circ\ \frac{1}{n}\sum_{\nu=1}^{n}|a_\nu|^2 = |c|^2 + \frac{1}{n}\sum_{\nu=1}^{n}|a_\nu - c|^2,$$

$$2^\circ\ \frac{1}{n}\sum_{\nu=1}^{n}|z-a_\nu|^2 = |z-c|^2 + \frac{1}{n}\sum_{\nu=1}^{n}|a_\nu - c|^2,$$

$$3^\circ\ \left|\frac{1}{n}\sum_{\nu=1}^{n} a_\nu\right|^2 \leqq \frac{1}{n}\sum_{\nu=1}^{n}|a_\nu|^2,$$

where in the last relation equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

§ 7. Sequences, Series, Products

7.1 Prove Lagrange's identity

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n}|a_k|^2 \sum_{k=1}^{n}|b_k|^2 - \sum_{1\leqq k<j\leqq n}|a_k\bar{b}_j - a_j\bar{b}_k|^2.$$

7.2 Prove the identities

$$\int_{-\infty}^{+\infty}(x^2+a^2)^{-n}\left\{x^n - \binom{n}{2}x^{n-2}a^2 + \binom{n}{4}x^{n-4}a^4 - \ldots\right\}dx = 0,$$

$$\int_{-\infty}^{+\infty}(x^2+a^2)^{-n}\left\{\binom{n}{1}x^{n-1} - \binom{n}{3}x^{n-3}a^2 + \binom{n}{5}x^{n-5}a^4 - \ldots\right\}dx = 0,$$

where $a > 0$ and $n > 1$ is a natural number.

7.3 Prove that:

1° $(1+z_n/n)^n \to 1$ as $n \to \infty$, if $z_n \to 0$ as $n \to \infty$,

2° $(1+z_n/n)^n \to e$ as $n \to \infty$, if $z_n \to 1$ as $n \to \infty$.

SOLUTION. The first assertion follows from the inequality

$$\left|\left(1+\frac{z_n}{n}\right)^n-1\right| = \left|1+\sum_{\nu=1}^{n}\binom{n}{\nu}\frac{z_n^\nu}{n^\nu}-1\right| \leqq \sum_{\nu=1}^{n}\binom{n}{\nu}\frac{|z_n|^\nu}{n^\nu}$$
$$= \left(1+\frac{|z_n|}{n}\right)^n-1.$$

If we put $z_n = 1+w_n$, then

$$\left(1+\frac{z_n}{n}\right)^n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{w_n}{n+1}\right)^n,$$

and hence 2° follows.

7.4 Determine the values of the sums

$$A = 1+\binom{n}{1}\cos\theta+\binom{n}{2}\cos 2\theta+\ldots+\binom{n}{n}\cos n\theta,$$
$$B = \binom{n}{1}\sin\theta+\binom{n}{2}\sin 2\theta+\ldots+\binom{n}{n}\sin n\theta.$$

SOLUTION. Since

$$A+iB = 1+\binom{n}{1}e^{i\theta}+\binom{n}{2}e^{2i\theta}+\ldots+\binom{n}{n}e^{ni\theta}$$
$$= (1+e^{i\theta})^n$$
$$= (1+\cos\theta+i\sin\theta)^n$$
$$= \left(2\cos^2\frac{\theta}{2}+2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$
$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^n$$
$$= 2^n\cos^n\frac{\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right),$$

we obtain

$$A = 2^n\cos^n\frac{\theta}{2}\cos\frac{n\theta}{2},$$
$$B = 2^n\cos^n\frac{\theta}{2}\sin\frac{n\theta}{2}.$$

7.5 Prove by use of the expansion of $(1+i)^n$ that

$$\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}=2^{n/2}\cos\frac{n\pi}{4},\qquad \sum_{k=0}^{[(n-1)/2]}(-1)^k\binom{n}{2k+1}=2^{n/2}\sin\frac{n\pi}{4}.$$

7.6 Find the sums

$$s_n=1+\frac{\sin x}{\sin x}+\frac{\sin 2x}{\sin^2 x}+\ldots+\frac{\sin nx}{\sin^n x},$$

$$\sigma_n=1+\frac{\cos x}{\sin x}+\frac{\cos 2x}{\sin^2 x}+\ldots+\frac{\cos nx}{\sin^n x},$$

when $\sin x\neq 0$.

SOLUTION. We have

$$\begin{aligned}\sigma_n+i(s_n-1)&=1+\frac{\cos x+i\sin x}{\sin x}+\frac{\cos 2x+i\sin 2x}{\sin^2 x}+\ldots\\&\qquad+\frac{\cos nx+i\sin nx}{\sin^n x}\\&=1+\frac{e^{ix}}{\sin x}+\frac{e^{2ix}}{\sin^2 x}+\ldots+\frac{e^{nix}}{\sin^n x}\\&=\frac{\dfrac{e^{(n+1)ix}}{\sin^{n+1}x}-1}{\dfrac{e^{ix}}{\sin x}-1}=\frac{1}{\sin^n x}\frac{e^{(n+1)ix}-\sin^{n+1}x}{e^{ix}-\sin x},\end{aligned}$$

and hence

$$\begin{aligned}\sigma_n&=\frac{1}{\sin^n x}\operatorname{Re}\left\{\frac{e^{(n+1)ix}-\sin^{n+1}x}{e^{ix}-\sin x}\right\}\\&=\frac{1}{2}\frac{1}{\sin^n x}\left\{\frac{e^{(n+1)ix}-\sin^{n+1}x}{e^{ix}-\sin x}+\frac{e^{-(n+1)ix}-\sin^{n+1}x}{e^{-ix}-\sin x}\right\}\\&=\frac{\sin^{n+2}x-\sin^{n+1}x\cos x-\sin x\cos(n+1)x+\cos nx}{\sin^n x(1-2\sin x\cos x+\sin^2 x)},\\s_n&=1+\frac{1}{\sin^n x}\operatorname{Im}\left\{\frac{e^{(n+1)ix}-\sin^{n+1}x}{e^{ix}-\sin x}\right\}\\&=\frac{2\sin^{n+2}x-2\sin^{n+1}x\cos x+\sin^n x-\sin(n+1)x\sin x+\sin nx}{\sin^n x(\sin^2 x-2\sin x\cos x+1)}.\end{aligned}$$

7.7 Determine the sum

$$S = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\cos k\theta}{(\cos \theta)^k},$$

using the identity

$$i \tan \theta \equiv -1 + \frac{e^{i\theta}}{\cos \theta}, \quad \cos \theta \neq 0.$$

ANSWER.

$$S = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^{n/2} \tan^n \theta, & \text{if } n \text{ is even.} \end{cases}$$

7.8 Prove that the Lambert series

$$\sum_{n=1}^{\infty} z^n/(1-z^n)$$

converges for $|z| < 1$ and diverges for $|z| \geqq 1$.

7.9 Prove that the function

$$f(z) = 1-(1-z) \exp\left(\frac{z}{1} + \frac{z^2}{2} + \ldots + \frac{z^k}{k}\right)$$

has the power series expansion

$$f(z) = b_{k+1}z^{k+1} + b_{k+2}z^{k+2} + \ldots,$$

where all the coefficients $b_{k+1}, b_{k+2}, \ldots$ are positive. (Cf. **7.20, 15.13**)

7.10 Check the expansion

$$\frac{1}{(z-a)(z-b)}$$

$$= \frac{1}{a-b}\left(\frac{1}{z} + \frac{a}{z^2} + \ldots + \frac{1}{b} + \frac{z}{b^2} + \ldots\right), \quad |a| < |z| < |b|,$$

$$= \frac{1}{a-b}\left(\frac{a-b}{z^2} + \frac{a^2-b^2}{z^3} + \ldots\right), \quad |z| > |b|,$$

$$a \neq 0, \qquad |a| < |b|.$$

HINT. Use the identities

$$\frac{1}{(z-a)(z-b)} \equiv \frac{1}{a-b}\left(\frac{1}{z-a}+\frac{1}{b-z}\right) \equiv \frac{1}{a-b}\left\{\frac{1}{z}\frac{1}{1-(a/z)}+\frac{1}{b}\frac{1}{1-(z/b)}\right\}$$

and

$$\frac{1}{(z-a)(z-b)} \equiv \frac{1}{a-b}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)$$
$$\equiv \frac{1}{a-b}\left(\frac{1}{z}\frac{1}{1-(a/z)}-\frac{1}{z}\frac{1}{1-(b/z)}\right)$$

for the cases $|a| < |z| < |b|$ and $|z| > |b|$, respectively.

7.11 Expand the function $1/\{(z-a)(z-b)\}$ in powers of $z-a$ and $z-b$.

7.12 Check the following expansions:

1° $$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}, \quad |z| < 2, \qquad \frac{1}{z-2} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}, \quad |z| > 2,$$

2° $$(z-1)^{-1}(z-2)^{-1} = \sum_{n=0}^{\infty}(1-2^{-n-1})z^n, \qquad |z| < 1,$$
$$= -\sum_{n=0}^{\infty}2^{-n-1}z^n - \sum_{n=1}^{\infty}z^{-n}, \qquad 1 < |z| < 2,$$
$$= \sum_{n=2}^{\infty}(2^{n-1}-1)/z^n, \qquad |z| > 2,$$

3° $$\frac{1}{(z-1)^2} = \sum_{n=0}^{\infty}(n+1)z^n, \qquad |z| < 1,$$
$$= \sum_{n=2}^{\infty}(n-1)z^{-n}, \qquad |z| > 1,$$

4° $$\frac{1}{(z-a)^m}$$
$$= (-a)^{-m}\left[1+\sum_{n=1}^{\infty}\frac{m(m+1)\ldots(m+n-1)}{n!}\frac{z^n}{a^n}\right], \quad |z| < |a|,$$
$$= \frac{1}{z^m}+\sum_{n=1}^{\infty}\frac{m(m+1)\ldots(m+n-1)}{n!}\frac{a^n}{z^{m+n}}, \qquad |z| > |a|,$$

m a natural number.

7.13 Expand the function $f(z) = 1/(1+z^2)$ in Taylor series about the points $z = 0$, $z = 1/2$, $z = -1$.

7.14 Check the following results:

$$1^\circ \quad \frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{(2k+1)!} z^{2k-1}, \qquad |z| > 0,$$

$$2^\circ \quad \frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \ldots, \quad 0 < |z| < 1,$$

$$3^\circ \quad \frac{1}{z^2(1-z)} = \sum_{k=0}^{\infty} z^{k-2}, \qquad 0 < |z| < 1,$$

$$= -\sum_{k=0}^{\infty} z^{-k-3}, \qquad |z| > 1,$$

$$4^\circ \quad \frac{z-1}{z^2} = \sum_{k=0}^{\infty} (-1)^k (k+1)(z-1)^{k+1}, \qquad |z-1| < 1.$$

7.15 Expand the function $f(z) = 1/\{(z-a)(z-b)(z-c)\}$, $|a| < |b| < |c|$ in Laurent series about various points.

7.16 Expand the functions $1/(z^2+a^2)$ and $1/(1+2z-z^2)$ in power series about various points and examine their convergence.

7.17 Expand the functions

$$1^\circ \quad \frac{1}{z^2+z+1}, \qquad 2^\circ \quad \frac{1}{e^z+1}, \qquad 3^\circ \quad \frac{1}{2\cosh z+1}$$

in power series about $z = 0$ and determine their radii of convergence.

7.18 Prove that the coefficients $a_0, a_1, a_2, \ldots$ in the expansion

$$\frac{1}{1-z-z^2} = a_0 + a_1 z + a_2 z^2 + \ldots$$

are the terms of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, \ldots$$

and that

$$a_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right\}.$$

7.19 Prove the identity

$$[\log (1+z)]^m = \sum_{n=m}^{\infty} \frac{m!}{n!} S_n^m z^n, \qquad |z| < 1,$$

where the Stirling numbers of the first kind S_n^m are defined by

$$x(x-1)(x-2)\ldots(x-n+1) = \sum_{m=1}^{n} S_n^m x^m,$$

$$S_0^0 = 1,\ S_n^0 = 0\ \ (n > 0),\ S_n^m = 0\quad (m > n \text{ or } m < 0).$$

HINT. Expand $[\text{Log}\,(1+z)]^m$ in a power series and show that its coefficients obey the same recurrence relation as the S_n^m.

7.20 Prove the relation

$$|E(z,\ p)-1| \leqq |z|^{p+1}, \qquad |z| \leqq 1,$$

where

$$E(z,\ p) = (1-z)\exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right).$$

(Cf. **7.9, 15.13**)

SOLUTION. The function $E(z,\ p)$ can be expanded in a series of the form

$$E(z,\ p) = 1 + \sum_{k=1}^{\infty} A_{kp} z^k, \tag{1}$$

where A_{kp}, $k = 1,\ 2,\ 3, \ldots$ are constants. Since

$$\frac{\mathrm{d}}{\mathrm{d}z} E(z,\ p) = -z^p \exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right),$$

the expansion of this derivative has the form

$$\frac{\mathrm{d}}{\mathrm{d}z} E(z,\ p) = -z^p \left(1 + \sum_{k=1}^{\infty} B_{kp} z^k\right), \tag{2}$$

where the coefficients B_{kp}, $k = 1,\ 2,\ 3, \ldots$, are positive numbers.

By (1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} E(z,\ p) = \sum_{k=1}^{\infty} k\, A_{kp} z^{k-1}.$$

Comparing this expression with (2), we find

$$A_{1p} = A_{2p} = \ldots = A_{pp} = 0, \quad A_{kp} < 0,\ k > p. \tag{3}$$

Hence we conclude from (1) that

$$|E(z,\ p)-1| = \left|\sum_{k=p+1}^{\infty} A_{kp} z^k\right|$$

$$\leqq \sum_{k=p+1}^{\infty} |A_{kp}|\,|z|^k = |z|^{p+1} \sum_{k=p+1}^{\infty} |A_{kp}|\,|z|^{k-p-1},$$

and that for $|z| \leqq 1$

$$|E(z, p)-1| \leqq |z|^{p+1} \sum_{k=p+1}^{\infty} |A_{kp}|. \tag{4}$$

Since $E(1, p) = 0$, we have by (1) and (3)

$$\sum_{k=p+1}^{\infty} A_{kp} = -1,$$

while

$$\sum_{k=p+1}^{\infty} |A_{kp}| = 1,$$

because $A_{kp} < 0$ for $k > p$. Hence it follows from (4) that $|E(z,p)-1| \leqq |z|^{p+1}$, $|z| \leqq 1$, p a natural number.

REMARK: See E. Hille, Analytic Function Theory, Vol. 1., New York, 1959, p. 257.

7.21 Prove the identity

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} \equiv \frac{n}{2^{n-1}}, \tag{1}$$

where $n > 1$ is a natural number.

SOLUTION. Since the zeros of the polynomial

$$\frac{x^n-1}{x-1} \equiv x^{n-1}+x^{n-2}+ \ldots +x+1, \qquad x \neq 1,$$

are

$$\varepsilon, \ \varepsilon^2, \ldots, \varepsilon^{n-1}, \quad \varepsilon = \cos(2\pi/n)+i \sin(2\pi/n),$$

we may write

$$x^{n-1}+x^{n-2}+ \ldots +x+1 = (x-\varepsilon)(x-\varepsilon^2) \ldots (x-\varepsilon^{n-1}).$$

Hence, letting $x = 1$, it follows that

$$n = (1-\varepsilon)(1-\varepsilon^2) \ldots (1-\varepsilon^{n-1})$$

and

$$n^2 = |1-\varepsilon|^2|1-\varepsilon^2|^2 \ldots |1-\varepsilon^{n-1}|^2. \tag{2}$$

Since

$$|1-\varepsilon^k|^2 = 1+(\varepsilon\bar{\varepsilon})^k-(\varepsilon^k+\bar{\varepsilon}^k) = 2\left(1-\cos\frac{2k\pi}{n}\right) = 4 \sin^2 \frac{k\pi}{n},$$

we conclude from (2) that (1) is valid.

7.22 Prove the identities

$$x^{2n+1}+1 \equiv (x+1)\prod_{k=1}^{n}\left(x^2+2x\cos\frac{2k\pi}{2n+1}+1\right),$$

$$x^{2n}-1 \equiv (x^2-1)\prod_{k=1}^{n-1}\left(x^2-2x\cos\frac{k\pi}{n}+1\right),$$

and hence derive the formulae

$$\prod_{k=1}^{n}\cos\frac{k\pi}{2n+1}\equiv\frac{1}{2^n},\quad \prod_{k=1}^{n}\sin\frac{k\pi}{2n+1}\equiv\frac{\sqrt{2n+1}}{2^n},$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{2n}\equiv\frac{\sqrt{n}}{2^{n-1}},\quad \prod_{k=1}^{n-1}\cos\frac{k\pi}{2n}\equiv\frac{\sqrt{n}}{2^{n-1}}.$$

HINT. For example, consider the second identity and write the left-hand side in the form

$$x^{2n}-1 = (x^2-1)(x^{2n-2}+x^{2n-4}+\ldots+x^2+1)=(x^2-1)g(x).$$

Find the roots of $g(x)$ and give x particular values.

7.23 Prove the relation

$$\prod_{k=1}^{n}\tan\frac{k\pi}{2n+1}=\sqrt{2n+1}.$$

7.24 Determine the value of the sum

$$n\cos\theta+(n-1)\cos 2\theta+\ldots+1\cos n\theta.$$

SOLUTION. We have

$$|z^n+z^{n-1}+\ldots+z+1|^2 = |(z^{n+1}-1)/(z-1)|^2,\quad z\neq 1.$$

Letting $z=\exp i\theta$, θ real, we obtain

$$\{e^{ni\theta}+e^{(n-1)i\theta}+\ldots+e^{i\theta}+1\}\{e^{-ni\theta}+e^{-(n-1)i\theta}+\ldots+e^{-i\theta}+1\}$$
$$=\frac{e^{(n+1)i\theta}-1}{e^{i\theta}-1}\cdot\frac{e^{-(n+1)i\theta}-1}{e^{-i\theta}-1}.$$

Hence

$$(n+1)+2\{n\cos\theta+(n-1)\cos 2\theta+(n-2)\cos 3\theta+\ldots+1\cos n\theta\}$$
$$=\sin^2\frac{n+1}{2}\theta\Big/\sin^2\tfrac{1}{2}\theta,\quad \sin\tfrac{1}{2}\theta\neq 0,$$

and

$$\sum_{k=1}^{n} k \cos (n-k+1)\theta = \frac{1}{2}\frac{\sin^2 \frac{n+1}{2}\theta}{\sin^2 \frac{1}{2}\theta} - \frac{1}{2}(n+1). \qquad (1)$$

Replacing n by $n-1$ and setting $\theta = 2\pi/n$, we obtain from (1)

$$(n-1)\cos\left\{1 \cdot \frac{2\pi}{n}\right\} + (n-2)\cos\left\{2 \cdot \frac{2\pi}{n}\right\} + \ldots + 1 \cos\left\{(n-1) \cdot \frac{2\pi}{n}\right\} = -\frac{n}{2}, \quad n > 1.$$

Remark 1. See G. Pólya-G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. II, 1925, p. 78, for another proof of (1).

Remark 2. On the basis of the identity

$$\frac{z^{2n}-1}{z+1} \equiv z^{2n-1}-z^{2n-2}+ \ldots +z-1, \qquad z \neq -1,$$

where $z = \exp i\theta$, θ real, one can prove the formula

$$(2n-1)\cos\theta-(2n-2)\cos 2\theta+ \ldots + 1 \cos (2n-1)\theta \equiv n - \frac{\sin^2 n\theta}{2\cos^2 \frac{1}{2}\theta}, \quad \cos\frac{1}{2}\theta \neq 0.$$

7.25 Prove

$$x^{2n}-2(\cos na)x^n+1 \equiv \sum_{k=1}^{n}\left[x^2-2x\cos\frac{na+2(k-1)\pi}{n}+1\right],$$

where n is a natural number and a is real. Hence evaluate the product

$$\prod_{k=1}^{n} \sin\left[\frac{a}{2}+\frac{(k-1)\pi}{n}\right].$$

§ 8. Analytic Functions and Their Properties

8.1 Let $f(z)$ be an analytic function of $z = \rho e^{i\theta}$. If u is a function of ρ only, then v is a function of θ only. Determine all functions with this property.

8.2 Show that, if $f(z)$ is an analytic function, the expressions

$$e^v(\cos u\,dx+\sin u\,dy),\quad e^v(\sin u\,dx-\cos u\,dy)$$

represent two total differentials.

8.3 Find the analytic function $f(z)$ such that

$$f(0)=0,\quad u(x,y)=\frac{x(1+x^2+y^2)}{1+2(x^2-y^2)+(x^2+y^2)^2}.$$

ANSWER. $f(z)=z/(1+z^2)$.

8.4 Is the function

$$w=|x^2-y^2|+2i|xy|$$

analytic in the domain $xy(x^2-y^2)>0$? Sketch this region.

8.5 Is there an analytic function $f(z)=u(x,y)+iv(x,y)$ satisfying the conditions

$u+v=(x-y)(\sin 2x-\cosh 2y)/(\cos 2x+\cosh 2y)$, $v(0,0)=0$?

8.6 Let $f(z)$ and $g(z)$ be two analytic functions. Show that $f(g)=f(g(z))$ is also an analytic function.

8.7 Determine A, B, C, D such that

1° $w=x^2+Axy+By^2+i(Cx^2+Dxy+y^2)$,

2° $w=\cos x\cosh y+A\cos x\sinh y$
$+i(B\sin x\sinh y+\sin x\cosh y)$,

are analytic functions and express these functions in terms of the variable z.

8.8 Show that every analytic function $f(z)$ satisfies the condition

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2.$$

8.9 If $f(z)$ is an analytic function, then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\log\{1+|f(z)|^2\}=\frac{4|f'(z)|^2}{\{1+|f'(z)|^2\}^2}.$$

8.10 Given an analytic function $f(z)$, check the relations

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2,$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|u|^p = p(p-1)|u|^{p-2}|f'(z)|^2.$$

8.11 Show that, if an analytic function $f(z)$ is real for every z, then it is a real constant.

8.12 Find an analytic function $f(z) = P+iQ$ which satisfies the relation

$$2xyP+(y^2-x^2)Q+2xy(x^2+y^2)^2 = 0. \tag{1}$$

SOLUTION. In polar coordinates $z = \rho e^{i\theta}$, the relation (1) assumes the form

$$P = Q \operatorname{cotan} 2\theta - \rho^4. \tag{2}$$

Since $f(z)$ satisfies the Cauchy-Riemann equations

$$\rho\frac{\partial P}{\partial \rho} = \frac{\partial Q}{\partial \theta}, \qquad -\rho\frac{\partial Q}{\partial \rho} = \frac{\partial P}{\partial \theta},$$

one has, by (2),

$$\frac{1}{\rho}\frac{\partial Q}{\partial \theta} = \frac{\partial Q}{\partial \rho}\operatorname{cotan} 2\theta - 4\rho^3, \quad -\rho\frac{\partial Q}{\partial \rho} = \frac{\partial Q}{\partial \theta}\operatorname{cotan} 2\theta - \frac{2}{\sin^2 2\theta}Q.$$

Eliminating $\partial Q/\partial\theta$, one finds now

$$\frac{\partial Q}{\partial \rho} - \frac{2}{\rho}Q = 2\rho^3 \sin 4\theta,$$

and hence

$$Q = \rho^2[C(\theta)+\rho^2 \sin 4\theta],$$

where $C(\theta)$ is an arbitrary differentiable function of θ.

Since Q is a solution of Laplace's equation, one obtains for C the differential equation

$$\frac{\mathrm{d}^2 C}{\mathrm{d}\theta^2} + 4C = 0,$$

whence

$$C = A \cos 2\theta + B \sin 2\theta$$

with A, B arbitrary real constants. Thus, (2) yields

$$P = \rho^2 \operatorname{cotan} 2\theta(A \cos 2\theta + B \sin 2\theta + \rho^2 \sin 4\theta) - \rho^4.$$

But P is also a solution of Laplace's equation, and therefore $A = 0$, B arbitrary.

Hence, finally,

$$\begin{aligned} P(\rho, \theta) &= \rho^2(B \cos 2\theta + 2\rho^2 \cos^2 2\theta) - \rho^4 \\ &= (x^2-y^2)[B+2(x^2-y^2)] - (x^2+y^2)^2, \\ Q(\rho, \theta) &= \rho^2 \sin 2\theta(B+2\rho^2 \cos 2\theta) = 2xy[B+2(x^2-y^2)], \end{aligned}$$

or

$$f(x+iy) = \{B(x^2-y^2)+y^4-6x^2y^2+x^4\} + i\{2Bxy+4xy(x^2-y^2)\}.$$

8.13 Determine the general form of the regular analytic function satisfying the relation

$$f(z)f(-z) = f(z^2) \tag{1}$$

inside the circle $|z| = 1$.

SOLUTION. If we put $z = 0$, we obtain from (1)

$$f(0)\{f(0)-1\} = 0, \text{ i.e., } f(0) = 1 \text{ or } f(0) = 0.$$

We shall examine first the case when $f(0) = 1$.

The function $f(z)$ has no zeros inside the circle $|z| = 1$. Indeed, if z_0 is a zero of $f(z)$, then all points $z_0^{2^n}$, n a natural number, are zeros of $f(z)$. But, since $|z_0| < 1$, it follows that

$$f(0) = \lim_{n\to\infty} f(z_0^{2^n}) = 0$$

which contradicts the hypothesis $f(0) = 1$.

Thus, we may assume that

$$f(z) = e^{u(z)}, \qquad u(0) = 0.$$

From (1), we obtain

$$u(z)+u(-z) = u(z^2). \tag{2}$$

Let

$$u(z) = \sum_{k=1}^{\infty} a_k z^k. \tag{3}$$

Substituting in (2) the corresponding expressions for $u(z)$ and $u(-z)$, we find

$$a_{2k} = a_k/2, \text{ and hence } a_{k\cdot 2^n} = a_k/2^n.$$

Therefore the series (3) can be written in the form

$$u(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_{(2n+1)2^k} z^{(2n+1)2^k} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{a_{2n+1}}{2^k} (z^{2n+1})^{2^k}$$

$$= \sum_{n=0}^{\infty} a_{2n+1} F(z^{2n+1}),$$

where

$$F(z) = \sum_{k=0}^{\infty} \frac{z^{2^k}}{2^k}.$$

Thus, the final result is

$$f(z) = \exp\left\{\sum_{n=0}^{\infty} a_{2n+1} F(z^{2n+1})\right\}.$$

Now, if $f(0) = 0$, we may put $f(z) = (-1)^k z^k g(z)$, $g(0) \neq 0$. Then (1) becomes

$$g(z)g(-z) = g(z^2),$$

and, since $g(0) \neq 0$, the solution of this functional equation is

$$g(z) = \exp\left\{\sum_{n=0}^{\infty} a_{2n+1} F(z^{2n+1})\right\}.$$

Thus, the general solution of (1) is

$$f(z) = (-1)^k z^k \exp\left\{\sum_{n=0}^{\infty} a_{2n+1} F(z^{2n+1})\right\}, \qquad k = 0, 1, 2, \ldots,$$

where the coefficients a_{2n+1} are such that the series (4) defines an analytic function inside the circle $|z| = 1$.

§ 9. Solutions of Differential Equations

9.1 Solve the differential equation

$$\frac{dy}{dx} = \frac{\operatorname{Re} z^n}{\operatorname{Im} z^n}, \qquad n \text{ an integer.}$$

SOLUTION. Letting $x = r\cos\theta$, $y = r\sin\theta$, we obtain

$$\frac{\sin\theta\, dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta} = \frac{r^n \cos n\theta}{r^n \sin n\theta},$$

i.e.,

$$\frac{dr}{r} = \frac{\sin(n+1)\theta}{\cos(n+1)\theta}\, d\theta,$$

and hence

$$r^{n+1}\cos(n+1)\theta = C, \quad C \text{ real.}$$

Thus, the final result is

$$\operatorname{Re} z^{n+1} = C.$$

9.2 Prove that the equations

$$\frac{dy}{dx} = \operatorname{cotan} v(x, y), \qquad \frac{dy}{dx} = -\tan v(x, y) \tag{1}$$

can be integrated by quadrature, if $v(x, y)$ is the imaginary part of an analytic function $f(z)$.

SOLUTION. If $v(x, y)$ is the imaginary part of an analytic function $f(z)$, then its real part can be determined from the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Since $\exp\{f(z)\}$ is analytic,

$$\frac{\partial}{\partial y}(e^u \cos v) = -\frac{\partial}{\partial x}(e^u \sin v), \quad \frac{\partial}{\partial y}(e^u \sin v) = \frac{\partial}{\partial x}(e^u \cos v), \tag{2}$$

and the equations (1) assume the form

$$e^u(\cos v\, dx - \sin v\, dy) = 0, \quad e^u(\sin v\, dx + \cos v\, dy) = 0.$$

By (2), these are exact differentials which can be integrated by quadrature, and $\exp(u)$ is an integrating factor for the equations

$$\cos v\, dx - \sin v\, dy = 0, \quad \sin v\, dx + \cos v\, dy = 0.$$

REMARK. See D. S. Mitrinović, Sur l'emploi de la partie réelle et de la partie imaginaire des fonctions analytiques dans l'intégration des équations differentielles, Tôhoku Mathematical Journal, **42**, 1936, p. 179—184.

9.3 Can a be given a value so that a solution $w(z)$ of the differential equation

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+z\frac{\mathrm{d}w}{\mathrm{d}z}-9\,w=e^z(z+a)$$

is a meromorphic function of z?

9.4 Determine a particular integral $w(z)$ of the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}z}+2zw+Ae^{z^2+Az}w^2=0,\quad \operatorname{Re}A>0,\ \operatorname{Im}A>0,$$

such that $w(z)\to\infty$ as $z\to 0$. Show that this integral has an infinite number of poles all of which lie on a straight line through the origin of the z-plane.

9.5 Find at the point $z=a$ the derivatives of the functions

$$\bar{z},\quad \operatorname{Re}z,\quad z\operatorname{Re}z,\quad |z|^2,\quad \sin\bar{z},\quad \cos\bar{z},$$

when z tends to a along a straight line determined by an angle θ. Is the derivative of any of these functions independent of θ? Are these functions analytic?

9.6 Show that Laplace's equation in two dimensions is satisfied by the function

$$v(x,\ y;\ t)=\frac{y}{(x-t)^2+y^2},$$

where t is a real parameter and $(x,\ y)\neq(t,\ 0)$. Determine an analytic function

$$f(z,\ t)=u(x,\ y;\ t)+iv(x,\ y;\ t)$$

such that $|f(z,\ t)|\to 0$ as $|z|\to\infty$.

9.7 Transform the expression $\Delta f=\partial^2 f/\partial x^2+\partial^2 f/\partial y^2$ by the substitution $x=u/(u^2+v^2)$, $y=v/(u^2+v^2)$, where u and v are new independent variables.

Prove by means of this result the proposition:

If the function $f(x, y)$ is a solution of Laplace's equation, then $f\{x/(x^2+y^2), y/(x^2+y^2)\}$ is also such a solution.

9.8 If $u(x, y)$, $f(x, y)$ are solutions of Laplace's equation in two dimensions, then the functions

$$u_x,\ xu_x+yu_y,\ yu_x-xu_y,\ f(u_x, u_y),\ x^2u_{xx}+2xyu_{xy}+y^2u_{yy}$$

are also such solutions.

9.9 If u is a harmonic function, then

$$\arctan\left(\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}\right)$$

is also a harmonic function.

9.10 Is there a function of the form

$$u = f\{(\cos 2x)/(\cosh 2y)\}$$

which satisfies Laplace's equation?

9.11 If the function $z(t) = x(t)+iy(t)$, where $x(t)$ and $y(t)$ are real functions of the real variable t, is differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re} z(t) = \operatorname{Re}\frac{\mathrm{d}z(t)}{\mathrm{d}t},\quad \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im} z(t) = \operatorname{Im}\frac{\mathrm{d}z(t)}{\mathrm{d}t}.$$

Using this fact determine the n-th derivatives of the functions

$$e^{pt}\cos qt,\qquad e^{pt}\sin qt,\qquad p,\ q \text{ real},$$

and

$$\frac{A}{(t-a)^2}+\frac{\bar{A}}{(t-\bar{a})^2} = 2\operatorname{Re}\frac{A}{(t-a)^2}.$$

9.12 Find the derivative of the function $f(z) = \bar{z}$ at the point $z = a$ in the direction of a smooth curve $x = g_1(t), y = g_2(t)$.

ANSWER: The required derivative is

$$\frac{\{g_1'(\tau)\}^2-\{g_2'(\tau)\}^2-2ig_1'(\tau)g_2'(\tau)}{\{g_1'(\tau)\}^2+\{g_2'(\tau)\}^2},$$

where τ is determined by the relation $a = g_1(\tau)+ig_2(\tau)$.

9.13 Given the differential equation

$$z^n \frac{\mathrm{d}^{2n} w}{\mathrm{d}z^{2n}} - a \frac{\mathrm{d}^n w}{\mathrm{d}z^n} = 0, \quad a \text{ real},$$

determine the general solution $w = w(z, a)$ of this equation and its singularities for $n = 3, 4$.

REMARK. See: D. S. Mitrinović-E. Grosswald: Solutions of a fourth-order differential equation, The American Mathematical Monthly, **68**, 1961, Problem 4932, p. 1013—1014.

9.14 Determine the general solution $w = w(z, a)$, a real, of the differential equation

$$z^{2n} \frac{\mathrm{d}^n w}{\mathrm{d}z^n} - aw = 0$$

and the singularities of w.

REMARK. See: E. Kamke, Differentialgleichungen, Vol. I., p. 540, 6th ed., Leipzig 1959.

§ 10. Conjugate Functions

10.1 Determine an analytic function $f(z)$ with the imaginary part $v = 1+x-2xy$.

10.2 Determine an analytic function $f(z)$ with the real part $u = (x+y)/(x^2+y^2)$.

ANSWER: $v = (x-y)/(x^2+y^2)+c$, c real, $f(z) = (1+i)/z+ic$.

10.3 Given $u(x, y) = e^x(x \cos y - y \sin y)$, find the conjugate function $v(x, y)$ for which $v(0, 0) = 0$ and express it in terms of z.

ANSWER. $v(x, y) = e^x(x \sin y + y \cos y)$, $f(z) = ze^z$.

10.4 Determine analytic functions $w = u+iv$ for which

1° u is a function of $x \cos a + y \sin a$, a real,

2° u^2+v^2 is a function of x,

3° u/v is a function of x.

10.5 Is there a real function $g(y)$, $g(0) = 0$, of a real variable y such that $u(x, y)=(1-x)/\{(1-x)^2+g(y)\}$ is the real part of an analytic function?

10.6 Let the function $f(x, y)$, $f(1, y) \neq 0$, have continuous second partial derivatives. Show that the function $f(x, y)/(1-x)$ cannot be the real part of an analytic function.

10.7 Prove that, if u and v are conjugate harmonic functions, then so are $au-bv$ and $av+bu$, a, b real.

10.8 If u and v are two conjugate harmonic functions, then so are the functions $u/(u^2+v^2)$ and $-v/(u^2+v^2)$.

10.9 Determine an analytic function $f(z)$ for which the real part $u(\rho, \theta)=(\cos\theta+\sin\theta)/\rho$, where $z = \rho e^{i\theta}$.

10.10 Determine all analytic functions $f(z)$ with real part $u(x, y) = g(x)+h(y)$, where $g(x)$ and $h(y)$ are real functions having first and second order continuous derivatives.

10.11 Determine an analytic function with the real part

$$\frac{x\cos y+y\sin y}{x^2+y^2}e^x.$$

10.12 Determine an analytic function $w = f(z)$ such that

$$|w|=(x^2+y^2)e^{xy}.$$

10.13 Determine all analytic functions $f(z) = u+iv$ the real parts of which are of the form $u = g(x)h(y)$, where $g(x)$ and $h(y)$ are two real functions having continuous first and second order derivatives.

SOLUTION. The function $u = f(x)g(y)$, as real part of an analytic function, must satisfy Laplace's equation. Hence

$$\frac{g''(x)}{g(x)} = -\frac{h''(y)}{h(y)}. \tag{1}$$

Since the variables x and y are independent, the left-hand side of this equation is independent of y and the right-hand side independent of x. Consequently, their common value cannot

depend on x or y, and must therefore be a real constant, say k. We must distinguish three cases:

1° $k = C^2$, $C \neq 0$ real.

Then we obtain from (1) the two equations
$g''(x) = C^2 g(x)$, $-h''(y) = C^2 h(y)$
with the general solutions
$g(x) = A' \sinh Cx + B' \cosh Cx$, $h(y) = D' \sin Cy + E' \cos Cy$,
where A', B', D', E' are real. Consequently, we have

$$u = (A' \sinh Cx + B' \cosh Cx)(D' \sin Cy + E' \cos Cy). \qquad (2)$$

Using (2) and the Cauchy-Riemann equations, we find then

$$v = (A' \cosh Cx + B' \sinh Cx)(-D' \cos Cy + E' \sin Cy) + \varphi(x). \qquad (3)$$

Substituting (2) and (3) in the second Cauchy-Riemann equation, we obtain

$$\varphi'(x) = 0, \quad \varphi(x) = C' = \text{const.}$$

Hence, finally,

$$f(z) = (A' \sinh Cx + B' \cosh Cx)(D' \sin Cy + E' \cos Cy) + i[C' + (A' \cosh Cx + B' \sinh Cx)(-D' \cos Cy + E' \sin Cy)].$$

2° $k = -C^2$, $C \neq 0$ real.

As before, we find, finally,

$$f(z) = (A'' \sin Cx + B'' \cos Cx)(D'' \sinh Cy + E'' \cosh Cy) + i[C'' + (A'' \cos Cx - B'' \sin Cx)(D'' \cosh Cy + E'' \sinh Cy)],$$

where A'', B'', C'', D'', E'' are real constants.

3° $k = 0$.

In this case we find from (1)

$$g''(x) = 0, \qquad h''(y) = 0,$$

and hence

$$g(x) = Ax + B, \quad h(y) = Dy + E,$$

where A, B, D, E are real constants. Hence

$$u = (Ax + B)(Dy + E),$$

$$\frac{\partial v}{\partial y} = A(Dy+E)$$

and, finally,

$$f(z)=(Ax+B)(Dy+E)+i[(C-BDx+AEy+\tfrac{1}{2}AD(y^2-x^2)].$$

§ 11. Transformation of Curves

11.1 Given the transformation

$$w = u(x, y)+iv(x, y) = z^2,$$

determine the curves in the w-plane which correspond to the straight lines $x = a$, $y = b$, where a, b are real.

HINT. The straight lines $x = a$, $y = b$ map into the sets of confocal, coaxial parabolas

$$u = a^2-v^2/(4a^2), \quad u = v^2/(4b^2)-b^2,$$

respectively, with vertices at $(a^2, 0)$, $(-b^2, 0)$. Since the function $w = z^2$ is analytic, the transformation will be conformal everywhere except at $z = 0$, where $dw/dz = 0$. Hence every coordinate rectangle in the z-plane without a vertex at the origin will be mapped into a curvilinear polygon with four vertices.

11.2 Given the transformation

$$w = z+1/z,$$

find the curves corresponding to

1° the circle $|z| = r$,

2° the straight line $y = mx$.

ANSWER. 1° The point w describes the ellipse

$$\frac{u^2}{(r+1/r)^2}+\frac{v^2}{(r-1/r)^2} = 1, \qquad r \neq 1.$$

11.3 Show that, when z describes the unit circle, the point $w = (az+b/z)/2$ describes an ellipse with axes $|a|+|b|$ and $||a|-|b||$ and foci determined by $w^2 = ab$, if $|a| \neq |b|$, while it describes a line segment through the origin if $|a| = |b|$.

11.4 Given the transformation

$$w = (a-be^z)/(1-e^z),$$

discuss the curves corresponding to the straight lines parallel to the coordinate axes in the z-plane.

11.5 Determine the curve which corresponds under the transformation

$$w = (z^2-az)/(az-1), \ a \text{ real},$$

to the unit circle $|z| = 1$.

ANSWER. Since $|z|^2 = z\bar{z}$, $\bar{w} = (\bar{z}^2-a\bar{z})/(a\bar{z}-1)$, we have

$$w\bar{w} = 1,$$

and hence the unit circles $|z| = 1$, $|w| = 1$ correspond.

REMARK. Discuss the case when a is imaginary.

11.6 Find the curve which corresponds under the transformation

$$w = (a+b)z^2+(a-b)/z^2, \quad a, \ b \text{ real},$$

to the unit circle $|z| = 1$.

11.7 Determine the curve which corresponds under the transformation

$$w = (1-iz)/(z-i)$$

to the x-axis of the z-plane.

11.8 Show that the point $w = (2+z)/(2-z)$ describes a circle in the positive direction as the point z moves along the y-axis in the positive direction.

11.9 Describe the path of the point

$$w = (z-1)/(z+1)$$

as z moves along the upper semi-circle from -1 to $+1$.

11.10 Given the transformations

$$w = az^2+2bz+c, \ w = a/(z-1)+b/(z+1)+c, \ a, b, c \text{ real},$$

discuss the curves corresponding to the circle $|z| = 1$.

11.11 Given the transformation

$$w = (z^2-1)/(z^2+1),$$

determine

1° the curve corresponding to the straight line $y = x$,

2° the curve corresponding to the straight line $y = -x$,

3° the region corresponding to the region $y^2-x^2 > 0$.

11.12 Discuss the curvilinear polygon corresponding to the square $|x|+|y| = 1$ under the transformation

$$w = e^z.$$

HINT. The transformation maps the straight lines $x = my+n$ into logarithmic spirals $\rho = e^n e^{m\theta}$.

11.13 Discuss the curve corresponding to the unit circle under the transformation

$$w = z(2z-1)/(2-z)$$

and sketch the regions corresponding to $|z| < 1$ and $|z| > 1$.

11.14 Determine the analytic functions $w = f(z)$ which map a curve C of the z-plane on a curve Γ of the w-plane, so that

1° the slope of the curve Γ at the point w_0 is equal to the angle between the tangent and the radius vector at the point z_0 of the curve C,

2° the angle between the tangent and the radius vector at w_0 of the curve Γ is equal to the angle between the tangent and radius vector at z_0 of the curve C,

3° the angle between the tangent and radius vector at w_0 of the curve Γ is equal to the slope of the curve C at z_0.

ANSWER. 1° $w = a \log z+b$, 2° $w = bz^a$, 3° $w = be^{az}$, $a > 0$.

11.15 Determine the curves on the Riemann sphere which under stereographic projection correspond to concentric circles in the complex plane.

SOLUTION. Let the point $N(X, Y, Z)$ on the Riemann sphere project into the point $M(x, y, 0)$. Since the straight line MN

passes through the North pole (0, 0, 1), it has the equations

$$\frac{x}{X} = \frac{y}{Y} = \frac{z-1}{Z-1}.$$

Hence, letting $z = 0$, we find

$$x = X/(1-Z),\ y = Y/(1-Z),\ z = 0.$$

This transformation maps the concentric circles

$$(x-a)^2+(y-b)^2 = R^2,\ z = 0$$

into

$$\left(\frac{X}{1-Z}-a\right)^2 + \left(\frac{Y}{1-Z}-b\right)^2 = R^2, \quad Z \neq 1,$$

$$X^2+Y^2+Z^2 = Z \tag{1}$$

on the Riemann sphere. Hence

$$2aX+2bY+(a^2+b^2-R^2-1)Z = a^2+b^2-R^2, \tag{2}$$

i.e., the curves are circles as intersections of the planes (2) and the sphere (1). Obviously, the distance of the plane (2) from the centre of the sphere (1), the origin, is less than $\frac{1}{2}$, so that the plane always intersects the sphere.

§ 12. Transformation of Domains

12.1 Determine the domain corresponding to $|x| < 1$, $|y| < 1$ under the transformation $w = 4iz+2-i$.

12.2 Which region corresponds to $\operatorname{Im} z \geqq 0$ for the transformation $w = (1-i)z$?

ANSWER. $\operatorname{Re} w+\operatorname{Im} w \geqq 0$.

12.3 Determine the region into which the transformation $w = z^2$ maps the region $1 \leqq |z| \leqq 2$, $-\pi/4 \leqq \arg z \leqq +\pi/4$.

ANSWER. $1 \leqq |w| \leqq 4$, $-\pi/2 \leqq \arg w \leqq \pi/2$.

12.4 Discuss the domain corresponding to $|z| \leqq 1$ under the transformation $w = 4/(z+1)^2$.

12.5 Determine the region corresponding to the inside of the triangle with vertices $z_1 = 0$, $z_2 = 1$, $z_3 = i$ under the transformation $w = z^2$.

12.6 Map the strip $0 < \operatorname{Re} z < \pi$, $\operatorname{Im} z > 0$ into the w-plane by means of the transformation $w = ik \operatorname{cotan} (z/2)$.

12.7 Map the region in the first quadrant bounded by $(x-a)^2+y^2 = a^2$, $(x-2a-b)^2+y^2 = b^2$, $(x-a-b)^2+y^2 = (a+b)^2$ by use of the transformation $w = 1/z$.

12.8 Show that the transformation $w = (z-i)^2/(z+i)^2$ maps the region $|z| \leqq 1$, $\operatorname{Re} |z| \geqq 0$ into $\operatorname{Im} w \geqq 0$.

12.9 Show that the transformation $w = \tan z$ maps the domain $-\pi/4 < \operatorname{Re} z < \pi/4$ into $|w| < 1$.

12.10 Discuss the region corresponding to $|z| \leqq 1$, $\operatorname{Re} z \geqq 0$, $\operatorname{Im} z \geqq 0$ under the transformation

$$f(z) = \frac{1}{\sqrt{1+z^2}}, \qquad f(0) = 1.$$

12.11 Is the region $\operatorname{Re} z \geqq 0$, $\operatorname{Im} z \geqq 0$ transformed into $|w| \leqq 1$ by the mapping $w = (z^2-i)/(z^2+i)$?

12.12 Which region corresponds to $\operatorname{Re} z > 2a$, $a \neq 0$, under the mapping $w = a^2/\bar{z}$?

12.13 Prove that the function $w = w(z) = z^2+2z+3$ defines a one-one transformation of the region $|z| < 1$.

PROOF. It is sufficient to show that $w(z_1) \neq w(z_2)$ for any two distinct points z_1 and z_2 lying inside the circle $|z| = 1$. Indeed, we have

$$\begin{aligned} |w(z_1)-w(z_2)| &= |z_1^2+2z_1-z_2^2-2z_2| \\ &= |z_1-z_2|\,|z_1+z_2+2| \geqq |z_1-z_2|(2-|z_1|-|z_2|). \end{aligned}$$

If $|z_1| < 1$, $|z_2| < 1$ and $z_1 \neq z_2$, then $|w(z_1)-w(z_2)| > 0$ and thus $w(z_1) \neq w(z_2)$.

12.14 Determine an analytic function $w = f(z)$, $f(0) = 1$, which transforms the region $|z| < 1$, $-\pi/3 < \arg z < \pi/3$ into the region $|w| < 1$.

12.15 Determine an analytic function $w = f(z)$, $f(0) = 0$ which maps the region $|z| < 1$, $0 < \arg z < \pi/2$ into the region $\operatorname{Im} w > 0$.

12.16 Determine the region which corresponds to the strip $-\pi < \operatorname{Im} z < \pi$ under the transformation $w = z + \exp z$.

12.17 Show that the mapping $w = \exp z$ transforms the rectangle $|x| \leqq \log 2$, $-\pi/4 \leqq y \leqq 3\pi/4$ into the semi-annular region shown in Fig. 6.

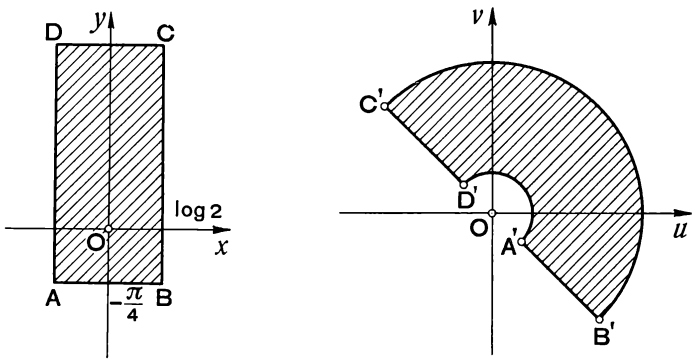

Fig. 6

12.18 Show that the function $w = \cosh z$ maps the rectangle with the vertices $z_1 = 0$, $z_2 = a$, $z_3 = a + (\pi/2)i$, $z_4 = (\pi/2)i$, $a > 0$, into a quadrant of an ellipse.

12.19 Map the following regions of the z-plane onto the Riemann sphere:

1° $|z| < 1$, 2° $|z| > 2$, 3° $1 < |z| < 2$, 4° $0 < \arg z < \pi/2$.

HINT. See Problem **11.15**.

12.20 Transform the region

$$|z| < 1, \quad \left| z - \frac{1}{2} \right| > \frac{1}{2}$$

into the w-plane by the mapping

$$w = \exp\left\{-\pi i \frac{z+1}{z-1}\right\}.$$

12.21 Transform the region

$$0 \leqq \operatorname{Re} z \leqq \pi, \quad 0 \leqq \operatorname{Im} z \leqq \log a$$

into the w-plane by means of the mapping

$$w = \frac{1}{a-e^{-iz}}, \qquad a > 1.$$

12.22 Show that

1° $w = z^2$ transforms the first quadrant of the z-plane into the domain $\operatorname{Im} w > 0$,

2° $w = e^z$ transforms the strip $0 < \operatorname{Im} z < \pi/2$ into the first quadrant of the w-plane,

3° $w = \cos z$ transforms the strip $-\pi/2 < \operatorname{Re} z < \pi/2$ into $\operatorname{Re} w > 0$,

4° $w = 1/(z+i)$ transforms the domain $y \geqq 0$ in the z-plane into $u^2+v^2+v \leqq 0$,

5° $w = (i-z)/(i+z)$ transforms the first quadrant of the z-plane into $|w| < 1$, $\operatorname{Im} w > 0$.

§ 13. The Bilinear Transformation

13.1 Given the bilinear transformation

$$w = a\frac{z+b}{z+c} \tag{1}$$

and the circle

$$|z-A| = R, \tag{2}$$

where $a \neq 0$, $c \neq b$ and $R > 0$ is a real constant, prove the following statements:

1° If $|A+c| \neq R$, then (1) transforms the circle (2) into a circle in the w-plane. Determine the centre and the radius of this circle and the region into which the interior of the circle (2) is mapped.

2° If $|A+c| = R$, then (1) transforms the circle (2) into a straight line in the w-plane. Find the equation of this straight line and the region into which (1) transforms the interior of the circle (2).

HINT. 1° The center A' and the radius R' of the circle in the w-plane are given by the formulae

$$A' = a\left\{1 + \frac{(b-c)(\bar{A}+\bar{c})}{|A+c|^2 - R^2}\right\}, \quad R' = \left|\frac{Ra(b-c)}{|A+c|^2 - R^2}\right|.$$

The z-plane is divided by the circle (2) into an internal and an external region. The region which contains the point $-c$ is transformed by (1) into the exterior of the image circle.

2° The equation of the straight line is

$$\left|w - a\frac{A+b}{A+c}\right| = |w-a|.$$

This straight line divides the w-plane into two half-planes. The exterior of the circle (2) is transformed by (1) into the half-plane containing the point a.

13.2 Determine the most general bilinear transformation which maps the circle $|z| = 1$ on the real axis of the w-plane.

ANSWER. The required transformation is

$$w = \frac{az + \bar{a}e^{i\alpha}}{z + e^{i\alpha}},$$

where $a \neq 0$ and α is real.

13.3 Given the transformation $w = (az+b)/(cz+d)$, $c \neq 0$, $ad-cb \neq 0$, examine the following problems:

1° What is the curve described by the point z as the point w moves along a circle or straight line through the point $w = a/c$? Show that the images of such circles and straight lines pass through the point $z = -d/c$.

2° Discuss the curves described by the point z as the point w moves along circles which do not pass through $w = a/c$.

3° If $a = d$, find the curve described by the point w as the point z moves along the circle $x^2+y^2+2x = 0$.

13.4 Given $w = (a+z)/(a-z)$, $a \neq 0$ real, show that the point w describes the circle $|w| = 1$ as z moves along the y-axis.

13.5 Show that $w = z/(1-z)$ maps the domain $\operatorname{Im} z \geqq 0$ into $\operatorname{Im} w \geqq 0$ and discuss the mapping of the circle $|z| = 1$, its inside and its outside into the w-plane.

13.6 Given the transformation $w = (z+2i)/(2iz-1)$, show that the circles $|w| = 1$ and $|z| = 1$ correspond.

13.7 Determine the most general transformation $w = (az+b)/(cz+d)$ which leaves the points z_1, z_2 fixed and find the image of the straight line through the points z_1, z_2.

13.8 Show that a bilinear transformation which maps the real axis into itself involves only real coefficients.

13.9 Given the bilinear transformation

$$w = a\frac{bz+1}{z+c}, \tag{1}$$

where $a \neq 0$, $bc \neq 1$, discuss the following questions:

1° Under what conditions will (1) transform the interior of the circle $|z| = 1$ into the interior of $|w| = 1$?

2° Under what conditions will the regions $|z| < 1$ and $|w| > 1$ correspond?

13.10 Given the transformation

$$w = (az+b)/(cz+d), \tag{1}$$

where a, b, c, d are real, discuss the following statements:

1° If $ad-bc > 0$, then the half-plane $y > 0$ is mapped by (1) into the half-plane $v > 0$.

2° If $ad-bc < 0$, then the half-plane $y > 0$ is mapped by (1) into the half-plane $v < 0$.

3° What curve in the w-plane corresponds to the circle $|z| = |d/c|$ and what regions correspond to the interior (exterior) of this circle?

13.11 Given the region bounded by the circles $|z-\alpha|=|\alpha-a|$, $|z-\beta|=|\beta-a|$, where $|\alpha-a|=|\alpha-\beta|+|\beta-a|$, discuss its transformation into the w-plane by means of the function $w = 1/(z-a)$.

13.12 Map the region $\operatorname{Im} z > 0$ into the w-plane by means of the transformation $w = e^{ai}(z-i)/(z+i)$, where a is real.

13.13 Determine the general form of the transformation $w = (az+b)/(cz+d)$ which maps the region $|z-i-1| \leqq 2$ into itself. When do such transformations leave the point $z = 3+7i$ fixed?

13.14 Find the region corresponding under the transformation $w = (az+b)/(cz+d)$, $ad-bc \neq 0$, to the region $\operatorname{Im}(z-z_1)/(z_2-z_1) > 0$, $z_1 \neq z_2$.

13.15 Determine a bilinear transformation which maps the circle $|z| = 2$ into the circle $|w+1| = 1$, the point $z = -2$ into the origin $w = 0$ and the origin $z = 0$ into the point $w = i$.

HINT. Use the fact that the cross-ratio

$$(z_1, z_2, z_3, z_4) = \frac{z_1-z_2}{z_1-z_4} : \frac{z_3-z_2}{z_3-z_4}$$

is invariant for bilinear transformations.

ANSWER: $w = -(z+2)/\{z(1+i)+2i\}$.

13.16 Given

$$z = a+r(i-t)/(i+t), \tag{1}$$

where $r > 0$, t real, find the locus of z as t varies through all real values.

13.17 Show that the point $w = (1-iz)/(z-i)$ moves along the semi-circle $|w| = 1$, $v \geqq 0$, as z moves from $(-1, 0)$ to $(1, 0)$ along the real axis.

13.18 Prove that the necessary and sufficient condition for $z = (aw+b)/(cw+d)$ to transform $|w| = 1$ into a straight line is $|d| = |c| > 0$.

13.19 Let z_1, z_2 be fixed points of

$$w = (az-\bar{b})/(bz+\bar{a}),$$

i.e., the roots of the equation

$$bz^2-(a-\bar{a})z+\bar{b} = 0.$$

Let the straight lines which join the North pole of the Riemann sphere to the points z_1, z_2 intersect the sphere at P_1, P_2, respectively. Prove that the segment P_1P_2 is a diameter of the Riemann sphere.

§ 14. Various Transformations

14.1 Let

$$w = (z^2-iz-1)/(2iz), \tag{1}$$

where $z = \exp(i\theta)$, θ real.

1° Show that w is a real function of θ.

2° Determine the curve described by w as θ varies from 0 to $\pi/2$.

HINT. 1° Since $|z| = 1$, $z\bar{z} = 1$, we find from (1)

$$w = y-\tfrac{1}{2},$$

and hence

$$x = \cos\theta,\ y = \sin\theta,\ u = y-\tfrac{1}{2},\ v = 0.$$

2° For $0 \leqq \theta \leqq \pi/2$, the relations

$$u = \sin\theta-\tfrac{1}{2},\quad v = 0$$

define the segment $[-\tfrac{1}{2}, \tfrac{1}{2}]$ of the u-axis.

14.2 Given the mapping

$$w = z^3-3z^2+3z+5,$$

determine

1° the angles of rotation and the magnifications at the points $z = 0$ and $z = i$,

2° the curves of constant angle of rotation,

3° the curves of constant magnification,

4° the points where the transformation is not conformal.

ANSWER: 1° 0, $3\pi/2$; 3,6; 2° $y = a(x-1)$, a real; 3° $(x-1)^2+y^2 = k^2$, k real; 4° $z = 1$.

14.3 Determine the region corresponding to the domain

$$|x+y| \leqq 1, \ |x-y| \leqq 1 \tag{1}$$

under the transformation

$$w = e^z.$$

Find the circumference and area of this region.

ANSWER. The image of the domain is shown in Fig. 7.

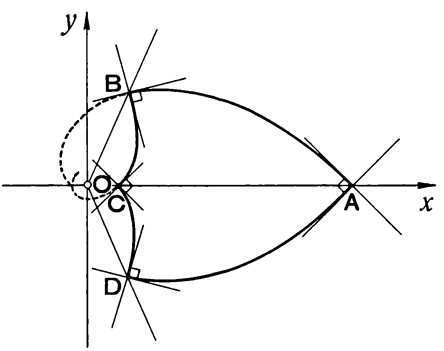

Fig. 7

Its boundary is formed by the curves

$$AB : \rho = e^{1-\theta}, \quad BC : \rho = e^{\theta-1}, \quad 0 \leqq \theta \leqq 1,$$
$$CD : \rho = e^{-\theta-1}, \quad DA : \rho = e^{\theta+1}, \quad -1 \leqq \theta \leqq 0,$$

which have the lengths $(e-1)\sqrt{2}$ and $(1-1/e)\sqrt{2}$, respectively; hence the total circumference of the region is $2(e-1/e)\sqrt{2}$ and its area $(e-1/e)^2/2$.

14.4 The function $w = \sqrt{z+1}$ maps the circle $|z| = 1$ into a closed curve. Determine the length s of this curve.

ANSWER: $s = B(1/4, 1/2)/\sqrt{2}$, where B is the Beta function.

14.5 Does the transformation

$$w = \frac{re^{i\theta}(z-a)}{r^2-(z-ir)(\bar{a}+ir)},$$

where $r > 0$ and θ are real, transform the region $|z-ir| \leqq r$ into the region $|w| \leqq 1$? Can one find a function $f(z)$ which maps the region $\operatorname{Im} z \geqq 0$ into $|w| \leqq 1$?

ANSWER. If $r \to \infty$, the required function becomes

$$f(z) = b\frac{z-a}{z-\bar{a}}, \quad |b| = 1.$$

14.6 Use the function

$$w = 4/(z+1)^2$$

to map the region

$$|z| < 1, \quad 0 < \arg z < \pi/2. \tag{2}$$

HINT. The corresponding regions are shown in Fig. 8, where the x-axis and u-axis correspond. Also, the y-axis becomes the parabola I ($\rho = 2/(1+\cos\theta)$, $w = \rho e^{i\theta}$) and the circle $|z| = 1$ the cardioid II ($\rho = 2(1+\cos\theta)$).

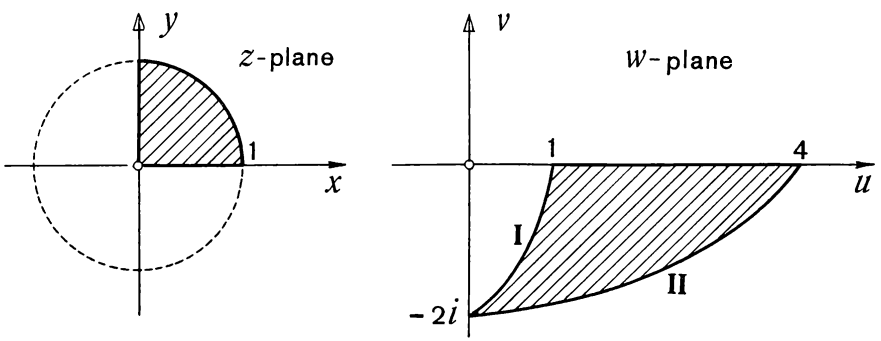

Fig. 8

14.7 Map each of the following regions by means of the given transformations:

1° $\operatorname{Im} z > 1$, $|z| < 2$, $\quad w = -\left(\dfrac{z+\sqrt{3}-i}{z-\sqrt{3}-i}\right)^3$;

2° $|z| > 2$, $|z-\sqrt{2}| < \sqrt{2}$, $\quad w = \left(\dfrac{z-\sqrt{2}(1-i)}{z-\sqrt{2}(1+i)}\right)^4$;

3° $|z| < 1$, $|z+i| > 1$, $\quad w = -\left(\dfrac{2z+\sqrt{3}+i}{2z-\sqrt{3}+i}\right)^3$;

$4°$ $|z|>1,\ |z+i|<1,$ $\quad w=\left(\dfrac{2z+\sqrt{3}+i}{2z-\sqrt{3}+i}\right)^3;$

$5°$ $|z|<2,\ 0<\arg z<\dfrac{1}{4}\pi,$ $\quad w=\left(\dfrac{z^4+16}{z^4-16}\right)^2;$

$6°$ $|z|<2,\ |z-1|>1,$ $\quad w=\exp\dfrac{2\pi iz}{z-2};$

$7°$ $\left|z-\dfrac{1}{2}i\right|>\dfrac{1}{2},\ \left|z-\dfrac{3}{2}i\right|>\dfrac{1}{2},$ $\quad w=\exp\dfrac{\pi iz}{z-i};$

$8°$ $|z|>2,\ |z-3|>1,$ $\quad w=\exp\left(\dfrac{2}{3}\pi i\dfrac{z-4}{z-2}\right);$

$9°$ $a<\operatorname{Re} z<b,$ $\quad w=\exp\left(\dfrac{\pi i}{b-a}(z-a)\right);$

$10°$ $\operatorname{Re} z>0,\ 0<\operatorname{Im} z<a,$ $\quad w=\left(\dfrac{e^{-\pi z/a}-1}{e^{-\pi z/a}+1}\right)^2.$

14.8 Determine a mapping which transforms the region $\pi/12\leqq\arg z\leqq\pi/6$ into the region $|w|\leqq 1$.

HINT. The given region is transformed by

$$w_1=ze^{-\frac{1}{12}\pi i}$$

into the region $0\leqq\arg w_1\leqq\pi/12$, which is transformed by $w_2=w_1{}^{12}$ into the region $\operatorname{Im} w_2\geqq 0$. Finally, $w=(w_2-i)/(w_2+i)$ transforms this region into the region $|w|\leqq 1$.

14.9 Transform the region

$$|z|\leqq r,\ \operatorname{Re} z\geqq 0$$

into the w-plane by means of the function

$$w=\frac{z^2-r^2+2rz}{z^2-r^2-2rz}.$$

HINT. Use the transformations

$$w_1=iz,\ w_2=\left(\frac{w_1+r}{w_1-r}\right)^2,\ w=i\frac{w_2-i}{w_2+i},$$

or

$$w_1 = \frac{1}{2r}z - \frac{r}{2z}, \quad w = \frac{w_1+1}{w_1-1}.$$

14.10 Discuss the transformation of the line segment

$$\operatorname{Re} z = p, \quad -\pi \leqq \operatorname{Im} z \leqq \pi$$

by means of the transformation

$$w = \frac{4e^z-3i}{5e^z+6i}.$$

14.11 Use the transformations

$$w_1 = z^3, \quad w_2 = \left(\frac{w_1+1}{w_1-1}\right)^2, \quad w = \frac{w_2-i}{w_2+i}$$

to map the sector

$$|z| \leqq 1, \quad 0 \leqq \arg z \leqq \pi/3$$

by means of the transformation

$$w = \frac{(1+z^3)^2-i(1-z^3)^2}{(1+z^3)^2+i(1-z^3)^2}.$$

14.12 Show that the transformation

$$w = (1+z)^2/(1-z)^2 \tag{1}$$

maps the region $|z| \leqq 1$, $\operatorname{Im} z \geqq 0$ into the region $\operatorname{Im} w \geqq 0$.

HINT. The transformation (1) is equivalent to the two transformations

$$w_1 = (1+z)/(1-z), \quad w = w_1^2.$$

The first of these maps the region $|z| \leqq 1$, $\operatorname{Im} z \geqq 0$ into $\operatorname{Re} w_1 \geqq 0$, $\operatorname{Im} w_1 \geqq 0$. Letting $w_1 = \rho e^{i\theta}$, it is readily seen that the second transformation completes the indicated mapping.

14.13 Discuss the mapping of the region $|z| < \frac{1}{2}$ by means of the transformation

$$w = (1-4z)^2/(1-z)^2.$$

14.14 Map the region

$$|z-1| \leqq \sqrt{2},\ |z+1| \leqq \sqrt{2}$$

into the w-plane by the transformation $w = 2z/(1-z^2)$.

14.15 Show that the transformation which leaves the origin fixed and the distance between any two points invariant is a rotation about the origin or a combination of such a rotation and a reflection in the x-axis.

SOLUTION. We require $w = f(z)$ such that

$$f(0) = 0, \tag{1}$$

$$|f(z_2)-f(z_1)|=|z_2-z_1| \tag{2}$$

for any z_1, z_2. Letting $z_1 = 0$, $z_2 = z$, we conclude from (1) and (2) that we must have

$$|f(z)|=|z|$$

for any value of z. Hence

$$f(z) = |z|\, e^{i\beta} = z\, e^{i(\beta-\theta)}, \tag{3}$$

where $\theta = \arg z$ and β depends on z. Let

$$z_1 = r_1 e^{i\theta_1},\ z_2 = r_2\, e^{i\theta_2},$$

and therefore

$$f(z_1) = r_1 e^{i\beta_1},\ f(z_2) = r_2 e^{i\beta_2},$$

so that

$$|f(z_2)-f(z_1)| = |r_2 e^{i\theta_2}-r_1 e^{i\theta_1}|,$$

i.e.,

$$\cos(\beta_2-\beta_1) = \cos(\theta_2-\theta_1),$$

provided $z_1 z_2 \neq 0$. Thus

$$\beta_2-\beta_1 = \pm(\theta_2-\theta_1)+2k\pi, \quad k = 0,\ \pm 1,\ \pm 2, \ldots,$$

or, using the minus sign ,

$$\beta_2+\theta_2 = \beta_1+\theta_1+2k\pi$$

and, in general,

$$\beta+\theta = \alpha+2k\pi, \tag{4}$$

where α is real. The final form of (3) is now

$$f(z) = \bar{z}e^{i\alpha} = \overline{ze^{-i\alpha}},$$

i.e., it is a rotation about the origin combined with a reflection in the x-axis.

Similarly, using the plus sign, we conclude that $f(z)$ is a rotation about the origin.

§ 15. Zeros and Singularities

15.1 Show that the function $f(z)=(z+a)\exp z+(z-a)\exp(-z)$, $a = \bar{a} > 0$, has only purely imaginary roots except for $z = 0$.

SOLUTION. If z is a zero,

$$|(z+a)e^z|=|(z-a)e^{-z}|. \tag{1}$$

But for $x > 0$, one has $e^x > e^{-x}$, $(x+a)^2 > (x-a)^2$, since $a > 0$, and hence

$$|(z+a)e^z| = e^x\sqrt{(x+a)^2+y^2} > e^{-x}\sqrt{(x-a)^2+y^2}=|(z-a)e^{-z}|,$$

while for $x < 0$, $e^x < e^{-x}$, $(x+a)^2 < (x-a)^2$, and hence

$$|(z+a)e^z| < |(z-a)e^{-z}|.$$

Thus, the equality (1) can only hold for $x = 0$ and the zeros of $f(z)$ must have the form iy, where y is given by

$$y+a\tan y = 0.$$

This equation has infinitely many real roots, and therefore the original function has one real and an infinite number of purely imaginary roots.

REMARK. Cf. the next problem.

15.2 Show that the equation

$$\tan z = az, \quad a \text{ real},$$

has an infinite number of real roots.

15.3 Given the function $G(z) = z/(z-1)$, examine the singularities of the functions $G(z)$, $\exp G(z)$, $\log G(z)$ and their nature at the point $z = \infty$.

15.4 Determine the zeros of the function $f(z) = \cos z - \cosh z$. Has this function multiple zeros?

SOLUTION. Since

$$\cos z - \cosh z = \cos z - \cos iz = -2 \sin\left(\frac{1+i}{2} z\right) \sin\left(\frac{1-i}{2} z\right),$$

the zeros of $f(z)$ are determined by the equations

$$\sin\left(\frac{1+i}{2} z\right) = 0, \quad \sin\left(\frac{1-i}{2} z\right) = 0.$$

Hence

$$z = k\pi(1 \pm i), \quad k = 0, \ \pm 1, \ \pm 2, \ldots,$$

and all the zeros lie on the straight lines $y = \pm x$, while the only double zero occurs at $z = 0$.

15.5 Find the zeros of the functions

$$\sin az - \sinh bz, \quad \cos az - \sinh bz.$$

15.6 Under what conditions does the function

$$f(z) = \sin pz - \cos qz$$

have double zeros and where do these zeros occur?

15.7 Show that the equation $\exp z = z$ has an infinite number of imaginary roots.

15.8 Solve the equation $\sin z = 2$.

15.9 Find the roots of the equation

$$\cos^2 z = (1+i)/2.$$

15.10 Solve for x the equation

$$\sum_{n=1}^{\infty} (xi)^{ni} = A,$$

where A is a constant. Has this equation real roots?

15.11 Prove that all the roots of the equation

$$\exp(1/z) = R \exp(i\varphi) \quad (R > 0 \text{ and } R \neq 1, \varphi \text{ real})$$

lie on the curve

$$x^2+y^2-\frac{x}{\log R}=0$$

and that the origin is a point of accumulation of these roots.

15.12 Solve for z the equations

1° $\exp\{(z+1)/z\}=1-i$, 2° $\tan z=e^{ki}$, k real.

15.13 Determine the coefficients $a_0, a_1, a_2, \ldots, a_k$ so that $z=0$ is a zero of order $k+1$ of the function

$$f(z)=1-(1-z)\exp(a_1z+a_2z^2+\ldots+a_kz^k).$$

ANSWER. $a_m=1/m$, $m=1, 2, \ldots, k$. (Cf. **7.9, 7.20**)

15.14 Let $f(z)$ and $g(z)$ be two meromorphic functions. Examine the nature of the functions

$$f(z)+g(z),\ f(z)g(z),\ f(z)/g(z),\ g(z)/f(z)$$

at a point a which is a pole of order m for $f(z)$ and a pole of order n for $g(z)$.

15.15 Let the point $z=a$ be a pole of the meromorphic function $f(z)$. Examine the nature of the function

$$\frac{f'''}{f''}+\frac{f'}{f}-2\frac{f''}{f'}$$

at the point $z=a$.

15.16 Let $P(z)$ and $Q(z)$ be two polynomials without common zeros. Determine the poles of the function

$$f(z)=\frac{1}{w}\frac{d^2w}{dz^2},\quad w=P(z)/Q(z).$$

If $P(z)$ has a zero of order k at $z=a$, what is the character of $f(z)$ at $z=a$?

15.17 Determine the singularities of the functions

$$\exp(z)/(z\sin(az)),\ z\exp(z)/\sin(az).$$

15.18 Given the functions

$$f(z)=(1-\cos z)/z^a,\ g(z)=(e^z-1-z)/z^a,$$

discuss their types of singularities for various values of the parameter a.

15.19 Discuss the singularities of the functions

1° $(1+\sqrt{z})^2+(1-\sqrt{z})^2$, 2° $\cos\sqrt{z}$,

3° $\dfrac{1}{1+\sqrt{z}}+\dfrac{1}{1-\sqrt{z}}$, 4° $\sqrt{z}\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}$.

NOTE. Explain the fact that the point $z=0$ is not a branch point of the functions 1°, 2°, 3°.

15.20 Prove that the equation $\exp z = az^n$, where n is a natural number and $a > e$, has n roots inside the circle $|z| = 1$.

§ 16. Residues

16.1 Determine the residues of the function

$$f(z) = 1/\{a+(az+b)\tan z\}^2$$

at its poles.

SOLUTION. Let θ be a pole of the function $f(z)$. Then θ is also a zero of the function $g(z) = a+(az+b)\tan z$.

Letting $z = t+\theta$, we find

$$g(t+\theta) = g'(\theta)t + \frac{g''(\theta)}{2!}t^2 + \ldots,$$

and hence

$$f(t+\theta)=1/\{g(t+\theta)\}^2 = \frac{1}{t^2}\frac{1}{\left\{g'(\theta)+\dfrac{g''(\theta)}{2}t+\ldots\right\}^2}$$

$$= \frac{1}{t^2\{g'(\theta)\}^2}\frac{1}{\left\{1+\dfrac{1}{2}\dfrac{g''(\theta)}{g'(\theta)}t+\ldots\right\}^2}$$

$$= \frac{1}{t^2\{g'(\theta)\}^2}\left\{1-\frac{1}{2}\frac{g''(\theta)}{g'(\theta)}t+\ldots\right\}^2$$

$$= \frac{1}{t^2\{g'(\theta)\}^2}\left\{1 - \frac{g''(\theta)}{g'(\theta)}t + \dots\right\}$$

$$= -\frac{g''(\theta)}{\{g'(\theta)\}^3}\frac{1}{t} + \frac{1}{\{g'(\theta)\}^2}\frac{1}{t^2} + \{A_0 + A_1 t + \dots\}.$$

Thus, the residue is $-g''(\theta)/\{g'(\theta)\}^3$. However, since

$$g'(z) = (az+b)(1+\tan^2 z) + a \tan z,$$
$$g''(z) = 2\{a + (az+b)\tan z\}(1+\tan^2 z) = 2g(z)(1+\tan^2 z),$$

we conclude that $g''(\theta) = 0$ and, consequently, that the residue at each pole of $f(z)$ is zero.

REMARK. Since

$$g''(z) - 2g'(z)\tan z = 2a,$$

the functions $g'(z)$ and $g''(z)$ have no common zeros, if $a \neq 0$. If $a = 0$, then $g(z) = b \tan z$.

16.2 Determine the residues of the function

$$F(z) = \frac{e^{1/z}z^n}{1+z}, \quad n \text{ a natural number},$$

at its poles and its essential singularities.

SOLUTION. The point $z = 0$ is an essential singularity and $z = -1$ is a first order pole of the function $F(z)$. Letting $z = 1/t$, we obtain

$$F(z) = F(1/t) = \frac{e^t}{t^{n-1}(1+t)},$$

whence it follows that the point $z = \infty$ is a pole of order $n-1$ $(n > 1)$. Forming the product of the series

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k}, \quad \frac{z^n}{1+z} = \sum_{k=0}^{\infty} (-1)^k z^{n+k},$$

we obtain the Laurent expansion of the function $F(z)$. It is convergent for $0 < |z| < 1$. The coefficient of $1/z$ is

$$\frac{1}{(n+1)!} - \frac{1}{(n+2)!} + \frac{1}{(n+3)!} - \dots,$$

i.e.,

$$\frac{1}{e}-\frac{1}{2!}+\frac{1}{3!}-\ldots+\frac{1}{n!} \quad \text{for odd } n, \tag{1}$$

$$-\frac{1}{e}+\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{1}{n!} \quad \text{for even } n, \tag{2}$$

and hence the residue of $F(z)$ at $z=0$ is given by (1) or (2). The residue at the pole $z=-1$ is $(-1)^n/e$.

Next, we will determine the residue at $z=\infty$. If

$$F(z)=F(1/t)=\sum A_k t^k,$$

then this residue is given by $(-A_{-1})$. Thus, forming the product of the series

$$e^t=\sum_{k=0}^{\infty}\frac{t^k}{k!}, \quad \frac{1}{t^{n-1}(1+t)}=\sum_{k=0}^{\infty}(-1)^k t^{-n+k+1},$$

we find

$$-A_{-1}=-\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!}=\operatorname*{Res}_{z=\infty}F(z).$$

16.3 Let $f(z)$ and $g(z)$ be two analytic functions and $f(a)\neq 0$. Show that,

1° if $z=a$ is a simple zero of $g(z)$, then

$$\operatorname*{Res}_{z=a}\{f(z)/g^2(z)\}=[f'(a)g'(a)-f(a)g''(a)]/g'^3(a),$$

2° if $z=a$ is a double zero of $g(z)$, then

$$\operatorname*{Res}_{z=a}\{f(z)/g(z)\}=[6f'(a)g''(a)-2f(a)g'''(a)]/[3g''^2(a)].$$

16.4 Determine the residue of the function

$$(z+a)e^z/z^4$$

at the pole $z=0$.

16.5 Find the residues of the function

$$g(z)=\frac{(c-a)^k(c-b)^l f(z)}{(z-a)^k(z-b)^l(c-z)}$$

at the poles a, b, where $f(z)$ is analytic, k and l are natural numbers and a, b, c are constants such that $f(a)f(b)f(c) \neq 0$.

16.6 Show that the function

$$\frac{1}{z}\left\{1+\frac{1}{z+1}+\frac{1}{(z+1)^2}+\ldots+\frac{1}{(z+1)^n}\right\}$$

has at the point $z=-1$ the residue $-n$.

16.7 Determine the poles of the function

$$(\operatorname{cotan}\pi z)/(z-a)^2$$

and the residues at these poles.

16.8 Determine the residues of the meromorphic function

$$(\sin z)^{-2}.$$

SOLUTION. The poles of the function $(\sin z)^{-2}$ are of second order and occur at the points $z_k = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. The Maclaurin expansion of this function is

$$\frac{1}{(\sin t)^2} = \frac{1}{(t-t^3/3!+\ldots)^2} = \frac{1}{t^2}\frac{1}{(1-t^2/3!+t^4/5!+\ldots)^2}$$
$$= \frac{1}{t^2}\left(1+\frac{t^2}{3!}+\ldots\right)^2.$$

Since $\sin^2(t+k\pi) = \sin^2 t$, this result represents the Laurent series of $f(t+k\pi)$. Hence the residues at all poles of this function are zero.

REMARK. These residues may also be evaluated by use of the formula

$$\lim_{z\to k\pi}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z-k\pi}{\sin z}\right)^2.$$

16.9 Let $f(z)$ be a meromorphic function and let z_1 and z_2 be two poles of $f(z^n)$, n a natural number, satisfying the relation

$$z_2 = z_1 \exp(2\pi i\nu/n) = z_1\varepsilon_n{}^{\nu},$$

where ν is an integer such that $1 \leqq \nu < n$. Prove the formula

$$\operatorname*{Res}_{z=z_2} f(z^n) = \varepsilon_n{}^{\nu} \operatorname*{Res}_{z=z_1} f(z^n).$$

16.10 Let

$$R(z) = A\{\prod_{\nu}(z-\zeta_\nu)^{p_\nu}\}/\{\prod_{\nu}(z-z_\nu)^{q_\nu}\}$$

be a rational function, where p_ν and $q_\nu > 0$ are integers. If the poles and zeros of $R(z)$ lie symmetrically with respect to the y-axis in such a manner that two symmetric poles and two symmetric zeros are of the same order, then

$$\operatorname*{Res}_{z=-\bar{z}_\nu} R(z) = -\frac{A}{\bar{A}}\overline{\operatorname*{Res}_{z=z_\nu} R(z)}.$$

GENERALIZATION. If the poles and zeros of $R(z)$ lie symmetrically with respect to a straight line in the same manner as above, then

$$\operatorname*{Res}_{z=z_2} R(z) = \frac{A}{\bar{A}}\, e^{2(k+1)\alpha i}\,\overline{\operatorname*{Res}_{z=z_1} R(z)}, \quad k = \sum_{\nu} p_\nu - \sum_{\nu} q_\nu,$$

where α is the angle of inclination of the straight line and z_1 and z_2 are symmetric.

16.11 Determine all the singularities of the function $f(z)=(2z+1)/\{1-\cos(z-a)\}$ and the residues at its poles.